

**HEURISTIC APPROACHES TO SOLVE RISK-ADJUSTED AND TIME-  
ADJUSTED DISCRETE ASSET ALLOCATION PROBLEM**

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## DECLARATION

I hereby declare that this thesis is my original work and it has

been written by me in its entirety. I have duly

acknowledged all the sources of information which have

been used in the thesis.

This thesis has also not been submitted for any degree in any

university previously.

A handwritten signature in black ink, consisting of three characters: '王' (Wang), '俊' (Jun), and '哲' (Zhe).

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WANG Junzhe  
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## Summary

This dissertation examines a real world private equity investment decision making process.

Private equity fund investments have the characteristics of high expected return, high risk and high illiquidity. Unlike traditional asset classes (eg. stocks and bonds), private equity investments are less flexible for the amount which is constrained by the target fund manager's requirement, and the investor's own diversification strategy. In this paper, the investment decision is simplified to a binary (discrete) problem ("yes" or "no") that is readily solvable with decision analysis tools.

The formulation of such a problem is a discrete asset allocation study. The nature of risk-adjusted investor utility behavior, as well as time-adjusted expected investment return complicate the problem to a Mixed Integer Non-Linear Programming (MINLP), for which there exists no efficient solving algorithms. Hence, several heuristic approaches are proposed to decompose the complex mathematical modeling into two sub-problems: 1) risk-adjusted Integer Quadratic Program (IQP), and 2) time-adjusted Non-Linear Program (NLP). In addition, comparisons are made among the heuristic approaches and exact approaches in terms of time efficiency and suboptimal level. The conclusion is that heuristic algorithms are much more time efficient than the exact approaches, and at the same time, they provide a satisfactory suboptimal solution. Lastly, a check-list table of different algorithms to use for solving different problem sizes is provided.

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# Chapter 1 Introduction

## 1.1 Background & Problem Description

This research work is motivated by a private equity fund investment decision problem faced by many fund of funds<sup>1</sup> managers.

Private equity, in finance, is an alternative asset class consisting of equity investment in operating companies that are not publicly traded on a stock exchange. It has the characteristics of greater expected return, higher risk and less liquidity when compared to traditional financial securities investments. Pension funds, sovereign wealth funds (SWF), insurance companies and high net worth individual (HNWI) are often attracted by this asset class because of its high yield, and also for the purpose of asset diversification. In recent years, for many headline successful companies, there are private equity players behind the scene; and FACEBOOK could be the most well known example.

Private equity fund of funds is an important player in private equity industry. Instead of investing directly in private companies, it invests in private equity funds to achieve a further risk diversification. Given a set of potential private equity fund candidates, the challenge facing a fund of funds manager is to identify the best private equity manager(s), which can produce the greatest return given the lowest risk.

From the risk point of view, the problem of diversification has been broadly looked into by numerous studies. Some well-known portfolio allocation strategies include “Markowitz efficient frontier”, “Mean-Variance Portfolio Theory”, “Capital Asset

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<sup>1</sup> A "fund of funds" (FOF) is an investment strategy of holding a portfolio of other investment funds rather than investing directly in shares, bonds or other securities. This type of investing activity is often referred to as multi-manager investment.



Pricing Model” (or CAPM), “Two Mutual Fund Theorem”, “Monetary Separation Theorem”, “Post-modern Portfolio Theory” (or PMPT) and “Black–Litterman Model”. However, they cannot be directly applied to our problem mainly due to its nature of high illiquidity (fixed holding period) and little flexibility in investment amount (discrete choices). For such a discrete decision problem, another frequently used technique is the decision analysis approach. However, due to the inter-correlation among different investment opportunities, the problem is modeled as an Integer Quadratic Program (IQP).

From the return point of view, time value consideration makes one investment less attractive when the holding period is longer than its alternative with the same level of absolute return. Hence, an additional decision variable comes into the picture and the decision maker has to choose the best holding period for its investments. This problem is easily solved given one single fund manager whose expected performance is a function of time; however, it is not trivial to choose one best common holding period for all the portfolio investments, which is a Non Linear Programming (NLP) problem.

In mathematical programming language, define the following variables

- $i$ : the  $i^{\text{th}}$  investment opportunity
- $N$ : total number of available investment opportunities
- $\delta_i \in \{0,1\}, i=1,\dots,N$ : indicator of whether investment opportunity  $i$  should be chosen
- $t \geq 0$ : holding period of the entire portfolio
- $T_i \geq 0$ : maximum value holding period where the expected return of investment

opportunity  $i$  can be improved to the maximum extent through the manager's operational value add (it can also be understood as the time period when the fund manager grows the company to a mature stage, and no more additional value can be created from the company)

- $a_i t^2 + b_i t + c_i$ : time-dependent expected return of investment opportunity  $i$  with  $a_i > 0$ ,  $b_i < 0$ ,  $-\frac{b_i}{2a_i} > 0$  and  $c_i \geq 0$ , so that the expected return is a decreasing function between  $[0, -\frac{b_i}{2a_i})$  and an increasing function between  $[-\frac{b_i}{2a_i}, +\infty)$ . The initial decreasing interval is due to some sunk cost such as transaction fees and due diligence expenses
- $\lambda_i(t)$ : indicator of whether portfolio holding period  $t$  is within or outside the maximum value holding period of investment opportunity  $i$ . And hence  $\lambda_{\{t \leq T_i\}} = 1$  if  $t \leq T_i$ ,  $\lambda_{\{t \leq T_i\}} = 0$  otherwise. Similarly  $\lambda_{\{t > T_i\}} = 1$  if  $t > T_i$ ,  $\lambda_{\{t > T_i\}} = 0$  otherwise.
- $d$ : discount rate of time value, or the risk free interest rate at which the amount will be compounded each period
- $r$ : Arrow-Pratt Coefficient of Absolute Risk Aversion
- $\rho_{ij}$ : correlation between two investment opportunities  $i$  and  $j$
- $\sigma_i$ : standard deviation of investment opportunity  $i$

The problem can be modeled as:

$$\underset{\{\delta_i\}_{i=1\dots N}, t}{\text{MAX}} \quad r \sum_{i=1}^N \frac{(a_i t^2 + b_i t + c_i) \lambda_{\{t \leq T_i\}} + (a_i T_i^2 + b_i T_i + c_i) \lambda_{\{t > T_i\}}}{(1+d)^t} \delta_i - \frac{r^2}{2} \sum_{i=1}^N \sum_{j=i}^N \rho_{ij} \sigma_i \sigma_j \delta_i \delta_j \quad \dots\dots (1.1)$$

There are two decision variables:

- $\delta_i \in \{0,1\}, i = 1, \dots, N$
- $t \geq 0$

The combination of risk-adjusted (IQP) and time-adjusted (NLP) considerations creates a Mixed Integer Non Linear Programming (MINLP) discrete asset allocation problem. To the best of the author's knowledge, there exists no efficient exact algorithm to solve such a problem.

## 1.2 Approach & Contribution

In this thesis, the complex MINLP model is decomposed into two sub-problems and the risk-adjusted and time-adjusted problems are solved separately.

To solve the risk-adjusted utility optimization problem, one starts with a single investment decision problem, where a standard decision analysis approach is applied to make the best choice between two investment candidates. the framework is restricted to constant absolute risk aversion (or “CARA”), and normally distributed returns. Due to the correlation between the current investment candidate and initial wealth, the Delta Property (the preference of a decision maker is independent of his/her initial wealth) no longer holds. Instead, the Relaxed Delta Property is proposed in this thesis, where the decision is independent of the expected return of the initial

portfolio. In the next step, single investment selection criteria are extended to multiple investment opportunities. It can be proven that the search for the optimal allocation is a power set problem and the complexity grows exponentially (i.e.  $O(C^n)$  in terms of Big-O notation, where  $c$  is a constant) with the number of potential opportunities. In this thesis, several heuristic algorithms are introduced to find the local optimal strategy (which stands a chance to be the global optimal solution) with a polynomial computation time (i.e.  $O(n^c)$  in terms of Big-O notation, where  $c$  is a constant).

To solve the time-adjusted expected return maximization problem, Matlab or CPLEX can be used with their self-embedded algorithms to find the solution readily.

Lastly, the optimization procedures for both sub-problems can be performed iteratively to keep improving the combined problem's result until neither sub-problems' solution can be further improved.

While the private equity investment decision is a real world problem, to the best of the author's knowledge, there is no literature on this topic. The major contribution of this thesis is to propose a number of heuristic approaches which solve this specific problem within reasonable time. In addition, a summary table of the best algorithms to use for different problem sizes is also presented.

### **1.3 Organization of Thesis**

This thesis is organized into six parts: Chapter 2 reviews the existing decision analysis and portfolio diversification techniques; Chapter 3 explores the decision process of a single investment opportunity and proposes a Relaxed Delta Property; Chapter 4 extends the single asset decision strategy to multiple investment opportunities and suggests several heuristic algorithms for the risk-adjusted asset allocation problem (IQP); Chapter 5 brings in the additional consideration of the time value of the expected return, and proposes to solve two sub-problems iteratively to find the optimal solution for the risk-adjusted and time-adjusted problem (MINLP); Chapter 6 summarizes the proposed approach's contributions and limitations, and also discusses the future work direction.

## Chapter 2 Literature Review

This thesis covers three topics, namely *Decision Analysis*, *Portfolio Diversification* and *Optimization Algorithm*.

### 2.1 Decision Analysis

Decision analysis (or “DA”) is the discipline for helping decision makers choose wisely under conditions of uncertainty (John, 2001). It is based on choosing the decision that maximizes the expected utility. Bernoulli (1713) proposed the concept of the expected utility model and Daniel (1738) developed the model further by solving the Petersburg paradox with the risk aversion assumption. Subsequently, von Neumann and Morgenstern (1944) formalized the expected utility theory and proposed the additive von Neumann–Morgenstern utility function. Following the work by Ramsey and von Neumann, Savage (1954) promoted subjective expected utility. Howard (1966) was the first person who brought up the term “decision analysis”. Arrow (1965) and Pratt (1964) defined the notions of constant absolute (or “CARA”) and constant relative risk aversion (or “CRRA”). Furthermore, they showed that linear and exponential utility functions are the only continuous utility functions with CARA property. In an earlier work, Pfanzagl (1959) proved that when the outcomes of a lottery are increased by a **Delta** amount, linear and exponential utility functions lead to an increase in the certainty equivalent of the lottery by the same **Delta** amount. Howard (1967) and Raiffa (1968) referred to this property as the “Delta Property”.

In recent work, led by Smith (1995), Nau (1995), McCardle (1998) and Copeland (2001), decision analysis is often integrated together with real options pricing

technique to value risks where the option and its underlying are not practically tradable, and forming a trading securities hedging portfolio is difficult, if not impossible.

## **2.2 Portfolio Diversification**

Modern Portfolio Theory (or MPT) is a theory to maximize portfolio expected return for a given risk, or equivalently minimize portfolio risk for a given level of expected return. Markowitz introduced this theory in a 1952 article and a 1959 book. MPT was further developed in the 1950s through the early 1970s, and there are many extensions since. Cohen & Pogue (1967), Arnott & von Gerneth (1983) and Goldfarb & Iyengar (2003) studied a systematic approach for asset allocation problems. Perold (1984), Tilley & Latainer (1985), Ghasemzadeh, Archer & Iyogun (1999) and Puelz (2002) proposed a series of different models for portfolio selection and optimization. Among all the research works, the ones that are most related to this paper should be Longstaff (2001) and Browne, Milevsky & Salisbury (2003), which studied the asset allocation strategy for illiquid assets. In addition, Patel & Subrahmanya (1982), Best & Hlouskova (2005), Kim & Viens (2010) and Sefton (2010) focused on the portfolio allocation problem with a fixed transaction cost.

As an application of portfolio selection and optimization, Perez & Malley (1983) used it for the social security system; Amit & Livnat (1989) applied it to corporate diversification; Kritzman (1992) and Gomes & Michaelides (2005) studied individual life-cycle asset allocation problem; Ankrum & Hensel (1993) proposed a commodity asset allocation solution; Eun & Resnick (1994) and Cavaglia & Moroz (2002) suggested an international cross-industry cross-country asset allocation strategy; and Chen, Ibbotson, Milevsky & Zhu (2006) found an application in life insurance.

## **2.3 Optimization Algorithm**

There are a number of algorithms designed for portfolio optimization problems.

One of the earliest studies was done by Kantorovich (1940) on Linear Programming, and then further developed by Dantzig (1947) for Simplex Method and Neumann (1947) for Theory of Duality.

Some major subfields of optimization programming include Integer Programming by Nemhauser & Wolsey (1988), Quadratic Programming by Murty (1988) and Nonlinear Programming by Bazaraa & Shetty (1979) and so on.

One important optimization technique is Heuristics Algorithm, which can provide approximate solutions to some optimization problems. Robin & Monro (1951) proposed Stochastic Optimization Methods. Matyas (1965) contributed his work on Random Optimization. Holland (1975) studied Genetic Algorithm. And Storn & Price (1997) proposed Differential Evolution Algorithm.

In particular, Hill Climbing Algorithm is one of the frequently-used Heuristics Algorithms. It is a popular mathematical optimization technique in computer science. Goldfeld, Quandt & Trotter (1966) studied this algorithm for a general optimizations problem; and Russell & Norvig (2003) provided a summary of various hill climbing techniques.



## 2.4 Research Gaps

Although there have been many research works published covering the above mentioned three topics, namely *Decision Analysis*, *Portfolio Diversification* and *Optimization Algorithm*, none of them can be directly applied to solve the “private equity fund investment decision problem”, described at the beginning of this thesis.

Firstly, *Decision Analysis*, although works to solve the discrete asset allocation problem, cannot be helpful to find the continuous optimal holding period solution.

Secondly, *Portfolio Diversification* takes both return and risk into account and is useful to model the problem in mathematical language. However, it does not assist to find the optimal solution and still cannot solve the problem with the best portfolio allocation choice.

In addition, *Optimization Algorithm* provides a list of tools that can be used to solve optimization problem. But it does not have a ready-to-use package for the problem in this thesis.

As a result, the proposed approaches in this thesis bridges the gaps among the above three topics and put them together to solve a specific problem in reality. It depends on *Decision Analysis* to make decision on individual investment selection; then model the “discrete and continuous” mixed problem in proper mathematical language based on *Portfolio Diversification* principles; and lastly solve the Mixed Integer Non Liner Programming (MINLP) problem with existing, but slightly modified *Optimization Algorithms*.

## Chapter 3 Single Investment Decision under Uncertain Wealth

### 3.1 Problem Description

In a decision making problem, there are a number of research works studying on utility functions with “Delta Property”, in which case the decision making is independent of the initial wealth. According to Clement and Reilly (2001), Delta Property is equivalent to Constant Absolute Risk Aversion (CARA).

However, none of the literature ever considers the case where the initial wealth is a random variable. This is what happens to a fund of funds investment decision making problem. A fund manager has an initial portfolio ( $W_0$ ) of  $N$  private equity funds, and needs to decide whether to include a new investment opportunity  $A$  into the portfolio. In addition, sometimes, the fund manager has to choose between two investment opportunities  $A$  and  $B$ . In the above two situations, not only should the uncertainty of initial portfolio be taken into account, but the correlation among  $W_0$ ,  $A$  and  $B$  should also be considered.

In such cases, the traditional Delta Property no longer holds; and this thesis proposes a Relaxed Delta Property to address this issue.

### 3.2 Constant Initial Wealth

This is the traditional case, where Delta Property, equivalent to Constant Absolute Risk Aversion (or “CARA”), guarantees the decision making is independent of initial wealth.

For a given utility function  $U(x)$ , the degree of absolute risk aversion is dependent on the wealth, which is often measured by *Arrow-Pratt Coefficient of Absolute Risk*

$$\text{Aversion: } r(x) = \frac{-u(x)''}{u(x)'}$$

Supposing  $r(x) = r$  is a constant, and Constant Absolute Risk Aversion (CARA) utility function has the form of either a linear or exponential function. Linear functions (risk neutral) are trivial cases, and this thesis focuses on the exponential form (risk averse or risk seek):  $U(x) = a - be^{-rx}$ , where  $r$  is the degree of absolute risk aversion.

- $a$  is a constant without sign restrictions.
- $b > 0$  and  $r > 0$  in the case of risk averse;  $b < 0$  and  $r < 0$  if risk seek.

Without loss of generality, this thesis assumes risk averse ( $b > 0$  and  $r > 0$ ). It can be shown that similar results, but with a sign alternation, apply to risk seek investors.

It has been proved that the Delta Property is equivalent to CARA:

$$A \succ_u B \Rightarrow A + w \succ_u B + w, \text{ independent of the constant value of } w$$

(" $\succ_u$ " refers to  $E[U(X_A)] > E[U(X_B)]$ )

### 3.3 Uncertain Initial Wealth + Zero Correlation

Consider the case that the initial wealth is a random variable with mean  $\mu_w$  and standard deviation  $\sigma_w$ , and there is no correlation between the new investment opportunities and the current wealth.

Let  $\rho_{X,Y} = 0$  be the correlation between two random variables  $X$  and  $Y$ .

Theorem 3.1:

$W$  is the initial wealth,  $A$  and  $B$  are two investment opportunities. If  $\rho_{W,A} = 0$ ,  $\rho_{W,B} = 0$ , and the utility function has the property of CARA; then the preference of a decision maker is independent of his initial wealth (Delta Property under uncertain initial wealth)

Proof:

$$\text{Given } A \succ_u B, \text{ i.e. } E[U(X_A)] = \sum_i p_i^A (a - be^{-ra_i}) > E[U(X_B)] = \sum_i p_i^B (a - be^{-rb_i})$$

$$\Rightarrow a - b \sum_i p_i^A e^{-ra_i} > a - b \sum_i p_i^B e^{-rb_i}$$

$$\Rightarrow \sum_i p_i^A e^{-ra_i} < \sum_i p_i^B e^{-rb_i} \quad \dots\dots (3.1)$$

$X_A$  is independent of  $W$

$$\Rightarrow \rho_{X_A, W} = 0 \quad \Rightarrow p_{i,j} = p_i \cdot p_j$$

$$\Rightarrow E[U(X_A + W)] = \sum_{i,j} p_{i,j} (a - be^{-r(a_i + w_j)}) = \sum_{i,j} p_i^A \cdot p_j (a - be^{-r(a_i + w_j)})$$

where  $p_{i,j} = P(X_A = a_i, W = w_j)$

$$\Rightarrow E[U(X_A + W)] = a - b \left( \sum_i p_i^A e^{-ra_i} \right) \left( \sum_j p_j e^{-rw_j} \right) \quad \dots\dots (3.2)$$

$$\text{Similarly } E[U(X_B + W)] = a - b \left( \sum_i p_i^B e^{-rb_i} \right) \left( \sum_j p_j e^{-rw_j} \right) \quad \dots\dots (3.3)$$

(3.1), (3.2) and (3.3) give the result:  $E[U(X_A + W)] > E[U(X_B + W)]$

$$\Rightarrow A + W \succ_u B + W \quad \text{Q.E.D}$$

Thus, Delta Property still holds even if initial wealth,  $W$ , is a random variable, provided that there is no correlation between  $W$  and the potential investment opportunities.

### 3.4 Uncertain Initial Wealth + Non-zero Correlation

#### 1) Counter Example of Delta Property

Below is an example in which Delta Property does not exist in the case where the initial portfolio is a random variable, and is correlated with the potential investment opportunities.

Example 3.1:

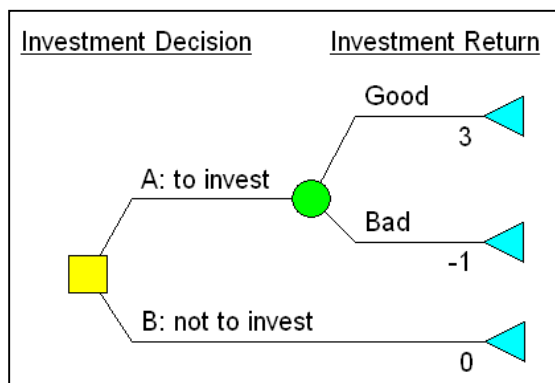


Figure 3.1 - Single Investment Decision Making without Initial Wealth

A risk-averse investor with zero initial wealth decides between two alternatives: he can either (choice A) invest with half chance to earn 3 and half chance to lose 1; or (choice B) not invest and get nothing. His utility function is given as

$$u(x) = 1 - e^{-(x+2)/2}$$

The result is to choose A, because  $E[u(X_A)] = 0.66 > E[u(X_B)] = 0.63$

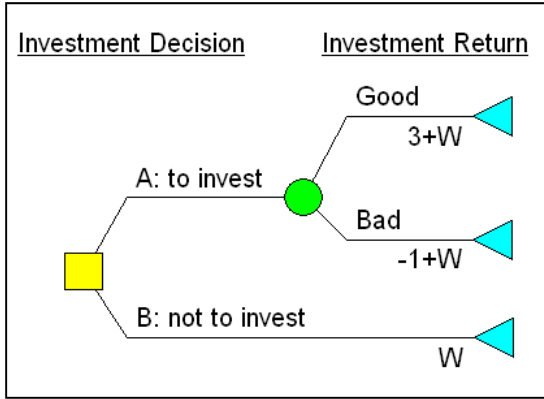


Figure 3.2 - Single Investment Decision Making under Uncertain Initial Wealth

Consider the same decision making problem as above, except that the decision maker has an initial wealth  $W \in \{5,0\}$  with

$$P(W = 5) = 0.5 \text{ and } P(W = 0) = 0.5$$

$$\text{Furthermore, } \rho_{A,W} = \frac{\text{Cov}(X_A, W)}{\sigma_A \cdot \sigma_{\tilde{W}_0}} = 1$$

In this case,  $E[u(X_A + W)] = 0.69 < E[u(X_B + W)] = 0.80$

In this example, despite the CARA utility function,  $A \succ_u B$  does not imply

$$A + W \succ_u B + W$$

## 2) Graphical Necessary Condition

With the failure of the Delta Property, a fast way is desired to make the decision whether or not to include an investment opportunity into the portfolio.

Start with a simple problem, to compare  $A + W$  vs.  $W$

The theorem below gives us a necessary condition of  $A + W \succ_u W$ .

To facilitate our discussion, for the rest of the thesis, unless specified, the utility function is CARA, and all the random variables are normally distributed.

Due to the above two assumptions, the expected utility is of the form:

$$E[u(X)] = a - b \cdot e^{r\mu_X - \frac{r^2\sigma_X^2}{2}} \quad \dots\dots (3.4)$$

The “iso-utility curve” is introduced here, which fixes

$$E[u(X)] = a - b \cdot e^{r\mu_X - \frac{r^2\sigma_X^2}{2}} = a - be^c \Rightarrow r\mu_X - \frac{r^2\sigma_X^2}{2} = c$$

$$\Rightarrow \mu_X = \frac{r\sigma_X^2}{2} + \frac{c}{r} \text{ is a quadratic function.}$$

Theorem 3.2:

Assume the initial portfolio  $W$  is characterized by  $(\sigma_0, \mu_0)$ , and the new portfolio with the inclusion of investment opportunity  $A$  is characterized by  $(\sigma_1, \mu_1)$ .

If  $A + W \succ_u W$ , then  $(\sigma_0, \mu_0)$  and  $(\sigma_1, \mu_1)$  fall in none of the three cases below:

$$1) \frac{\mu_1 - \mu_0}{\sigma_1 - \sigma_0} \leq r\sigma_0, \mu_1 - \mu_0 > 0 \text{ and } \sigma_1 - \sigma_0 > 0 \quad \dots\dots (3.5)$$

$$2) \mu_1 - \mu_0 > 0 \text{ and } \sigma_1 - \sigma_0 < 0 \quad \dots\dots (3.6)$$

$$3) \frac{\mu_1 - \mu_0}{\sigma_1 - \sigma_0} \geq r\sigma_0, \mu_1 - \mu_0 < 0 \text{ and } \sigma_1 - \sigma_0 < 0 \quad \dots\dots (3.7)$$

Proof:

Note that  $\frac{\mu_1 - \mu_0}{\sigma_1 - \sigma_0}$  is the gradient of the straight line passing through  $(\sigma_0, \mu_0)$  and  $(\sigma_1, \mu_1)$  ; and  $\frac{\partial(\frac{r\sigma^2}{2} + \frac{c}{r})}{\partial\sigma} \Big|_{\sigma=\sigma_0} = r\sigma_0$  is the tangent line of  $\mu = \frac{r\sigma^2}{2} + \frac{c}{r}$  at  $(\sigma_0, \mu_0)$

The proof of this theorem is intuitive.  $A+W \succ_u W$  implies that the new portfolio  $(\sigma_1, \mu_1)$  must be above the quadratic curve.

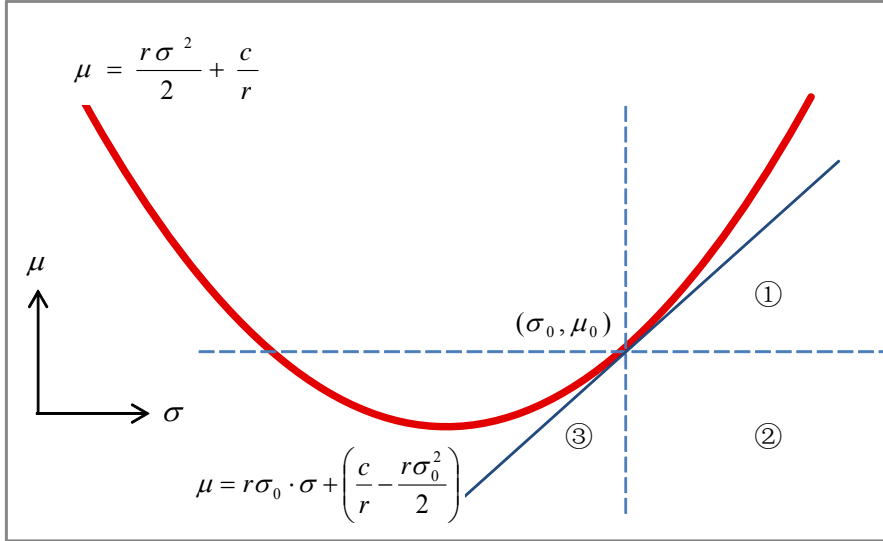


Figure 3.3 - Necessary Condition for Including an Investment Opportunity

Graphically, the above mentioned three conditions are the regions 1, 2, 3 respectively, where AB is the tangent line at  $(\sigma_0, \mu_0)$ . As noticed, all the three areas are below the quadratic curve, and hence the new portfolio is inferior than the original one.

Q.E.D

In other words, the Theorem is saying that if any of the above three conditions is met,  $A+W \prec_u W$  and  $A$  should not be added into the portfolio.



As noticed, Theorem 3.2 is only a necessary condition for  $A+W \succ_u W$  and the sufficient condition is discussed below.

### 3) Graphical Sufficient Condition

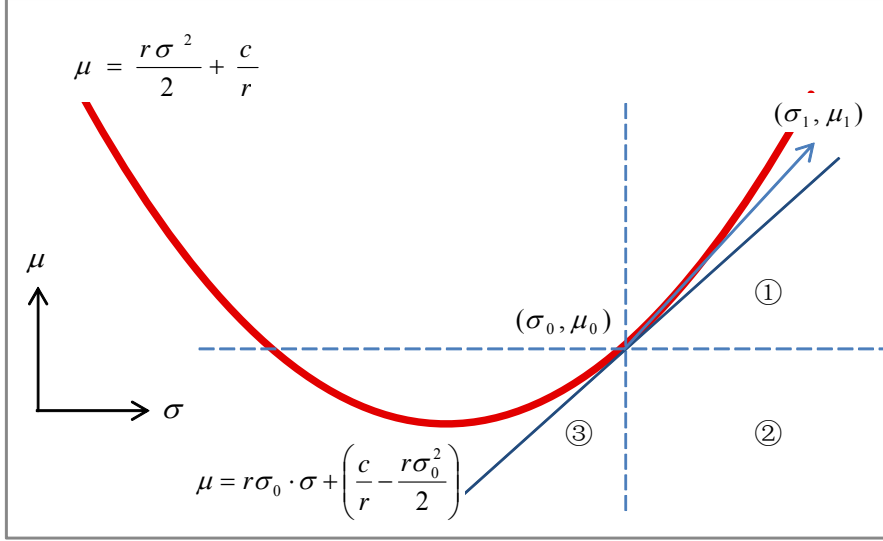


Figure 3.4 - Sufficient Condition for Including an Investment Opportunity

Even if  $(\sigma_1, \mu_1)$  is out of the areas 1, 2, 3, it still can be inferior than  $(\sigma_0, \mu_0)$ , as shown in Figure 3.4

Hence, in order that  $(\sigma_1, \mu_1)$  is above the quadratic curve, the vector defined by  $(\sigma_0, \mu_0)$  and  $(\sigma_1, \mu_1)$  must intersect with the curve once and only once, at  $(\sigma_0, \mu_0)$ .

### 4) Relaxed Delta Property

#### Theorem 3.3:

Assume a CARA utility function with constant Arrow-Pratt Coefficient of Absolute Risk Aversion  $r$ ;  $A, B$  are two investment opportunities, and  $W$  is the initial wealth.

$A, B$  and  $W$  follow normal distribution with below parameters:

$$X_A \sim N(\mu_A, \sigma_A), X_B \sim N(\mu_B, \sigma_B), W \sim N(\mu_W, \sigma_W)$$

Furthermore,  $\rho_{A,W} = \rho_A$  and  $\rho_{B,W} = \rho_B$  are the correlations of  $A, B$  and  $W$  respectively.

$$\text{Let } (r\mu_A - \frac{r^2\sigma_A^2}{2}) - (r\mu_B - \frac{r^2\sigma_B^2}{2}) = \xi \quad \dots\dots (3.8)$$

In the case without the consideration of the initial wealth (local property):

$\xi > 0 \Rightarrow A$  is preferred over  $B$  ;

$\xi = 0 \Rightarrow$  both  $A$  and  $B$  are equivalent;

$\xi < 0 \Rightarrow B$  is preferred over  $A$  ;

In the case with the consideration of the initial wealth (global property):

$$\rho_A\sigma_A - \rho_B\sigma_B > \frac{\xi}{r^2\sigma_W} \Rightarrow A \text{ is preferred over } B ;$$

$$\rho_A\sigma_A - \rho_B\sigma_B = \frac{\xi}{r^2\sigma_W} \Rightarrow , \text{ both } A \text{ and } B \text{ are equivalent;}$$

$$\rho_A\sigma_A - \rho_B\sigma_B < \frac{\xi}{r^2\sigma_W} \Rightarrow B \text{ is preferred over } A ;$$

Proof:

(Local Property)

$$E[u(X)] = E[a - be^{-rX}] = a - be^{-r\mu_X + \frac{r^2\sigma_X^2}{2}} \quad \dots\dots (3.9)$$

Because of the property:  $X \sim N(\mu_X, \sigma_X) \Rightarrow E[e^X] = e^{\mu_X + \frac{\sigma_X^2}{2}}$

$$\Rightarrow E[u(X_A)] - E[u(X_B)] = a - be^{-r\mu_A + \frac{r^2\sigma_A^2}{2}} - (a - be^{-r\mu_B + \frac{r^2\sigma_B^2}{2}})$$

$$\text{So, } r\mu_A - \frac{r^2\sigma_A^2}{2} > r\mu_B - \frac{r^2\sigma_B^2}{2} \Rightarrow E[U(X_A)] > E[U(X_B)]$$

$$\text{As a result, } \xi = (r\mu_A - \frac{r^2\sigma_A^2}{2}) - (r\mu_B - \frac{r^2\sigma_B^2}{2}) > 0 \Rightarrow E[U(X_A)] > E[U(X_B)]$$

This proves the local property

(Global Property)

$$X_A + W \sim N(\mu_A + \mu_W, \sigma_{A+W}) \text{ where } \sigma_{A+W} = \sqrt{\sigma_A^2 + \sigma_W^2 + 2\rho_A\sigma_A\sigma_W} \quad \dots\dots (3.10)$$

$$(3.9) \text{ and } (3.10) \text{ imply } E[U(X_A+W)] = a - be^{-r(\mu_A+\mu_W) + \frac{r^2(\sigma_A^2+\sigma_W^2+2\rho_A\sigma_A\sigma_W)}{2}} \quad \dots\dots (3.11)$$

Comparison between  $E[U(X_A + W)]$  and  $E[U(X_B + W)]$  is to compare:

$$r(\mu_A + \mu_W) - \frac{r^2(\sigma_A^2 + \sigma_W^2 + 2\rho_A\sigma_A\sigma_W)}{2} \text{ vs. } r(\mu_B + \mu_W) - \frac{r^2(\sigma_B^2 + \sigma_W^2 + 2\rho_B\sigma_B\sigma_W)}{2}$$

$$[r(\mu_A + \mu_W) - \frac{r^2(\sigma_A^2 + \sigma_W^2 + 2\rho_A\sigma_A\sigma_W)}{2}] - [r(\mu_B + \mu_W) - \frac{r^2(\sigma_B^2 + \sigma_W^2 + 2\rho_B\sigma_B\sigma_W)}{2}]$$

$$= (r\mu_A - \frac{r^2\sigma_A^2}{2}) - (r\mu_B - \frac{r^2\sigma_B^2}{2}) - r^2\rho_A\sigma_A\sigma_W + r^2\rho_B\sigma_B\sigma_W = \xi - r^2\sigma_W(\rho_A\sigma_A - \rho_B\sigma_B)$$

$$\text{Therefore, } \rho_A\sigma_A - \rho_B\sigma_B < \frac{\xi}{r^2\sigma_W} \Rightarrow E[U(X_A + W)] > E[U(X_B + W)] \quad \dots\dots (3.12)$$

This proves the global property.

Q.E.D

The *Table 3.1* below summarizes Theorem 3.3:

*Table 3.1 - Choice between Two Opportunities under CARA Utility Function*

Local Problem	$\xi > 0$	$A \succ_u B$
	$\xi = 0$	$A =_u B$
	$\xi < 0$	$A \prec_u B$
Global Problem	$\rho_A \sigma_A - \rho_B \sigma_B < \frac{\xi}{r^2 \sigma_W}$	$A + W \succ_u B + W$
	$\rho_A \sigma_A - \rho_B \sigma_B = \frac{\xi}{r^2 \sigma_W}$	$A + W =_u B + W$
	$\rho_A \sigma_A - \rho_B \sigma_B > \frac{\xi}{r^2 \sigma_W}$	$A + W \prec_u B + W$

There are two things to note from this summary table:

- Delta property no longer exists for CARA utility functions

Assuming  $\xi = 0$  (decision maker is indifferent between A and B for a local problem); the global optimal choice depends on the size of  $\rho_A \sigma_A - \rho_B \sigma_B$ :

- The smaller  $\rho_B$  is, the more likely B is chosen

The smaller  $\sigma_B$  is, the more likely B is chosen

This result is consistent with intuition: smaller  $\sigma_B$  is implying that B is less

risky and hence more appealing for a risk averse decision maker; smaller  $\rho_B$  is implying that by adding this asset into the portfolio, part of the risk is diversified away, and hence is preferred by the investor.

Furthermore, the global problem's decision criteria is independent of  $\mu_W$ , the expected value of initial wealth. It is named as Relaxed Delta Property, which is defined below:

*Definition 3.1:*

Given an uncertain initial portfolio  $W$ , the preference of the decision maker has Relaxed Delta Property, if its decision making process is independent of the expected return of initial portfolio,  $\mu_W$ .

*Theorem 3.4:*

CARA utility function + Normal distribution implies Relaxed Delta Property.

*Proof:*

This is a direct result of *Theorem 3.3* and *Definition 3.1* Q.E.D

### **3.5 Case Study – Investment Decision between Two Opportunities**

*Example 3.2:*

Let  $N(\mu, \sigma)$  represent a random variable with normal distribution of expected return  $\mu$ , and standard deviation (or volatility, i.e. proxy for risk level)  $\sigma$ .

Suppose a decision maker with CARA utility function ( $r = 2$ ) has initial portfolio  $W$  characterized by  $W \sim N(20, 4)$ .

He is to choose between investment opportunities  $A \sim N(8,2)$  and  $B \sim N(1,3)$  ;

furthermore, the correlations are  $\rho_A = 0.1$  and  $\rho_B = -0.5$

$$\xi = (r\mu_A - \frac{r^2\sigma_A^2}{2}) - (r\mu_B - \frac{r^2\sigma_B^2}{2}) = 24, \quad \frac{\xi}{r^2\sigma_W} = \frac{24}{2^2 \times 4} = 1.5 > 0 \Rightarrow A \succ_u B$$

$$\rho_A\sigma_A - \rho_B\sigma_B = 1.7$$

$$\Rightarrow \rho_A\sigma_A - \rho_B\sigma_B > \frac{\xi}{r^2\sigma_W}$$

$$\Rightarrow W + A \prec_u W + B \Rightarrow B \text{ is chosen}$$

This result seems to counter intuition, because  $A$  has higher expected return and lower risk than  $B$  on a standalone basis. However,  $B$  is negatively correlated with the initial wealth, and the combined effect of  $B$  and  $W$  is less risky than the combined effect of  $A$  and  $W$ .

The solution can be verified graphically. As in *Figure 3.5*, the iso-utility curve of  $W + B$  is above that of  $W + A$ . Hence  $B$  is preferred over  $A$ .

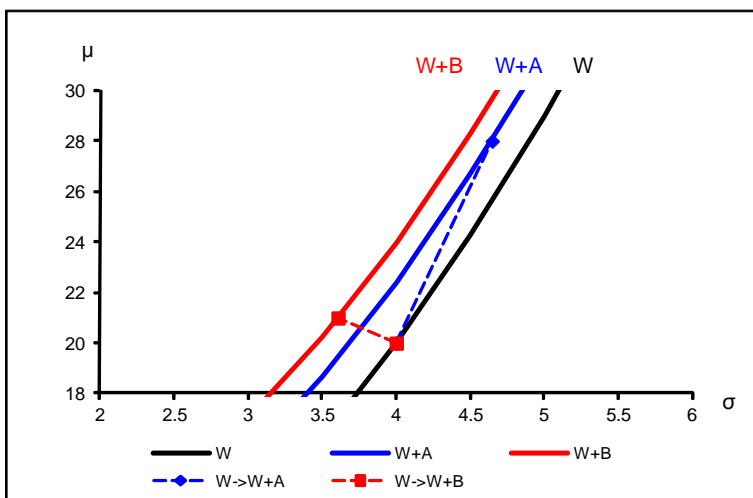


Figure 3.5 - Application of Relaxed Delta Property

## Chapter 4 Risk-Adjusted Multiple Investment Decisions

### 4.1 Problem Description

In the previous chapter, problem of how to make a choice among two options has been studied. However, in most cases, a fund of funds manager is presented with  $N$  (instead of only two) investment opportunities, out of which he should select a subset to maximize the expected utility value.

Given  $W$  is the initial wealth and  $\Omega$  is a set of possible investment opportunities, define  $W + \Omega$  as  $W + \sum_{A_i \in \Omega} A_i$ .

So the problem can be expressed as

With  $N$  potential investment opportunities,  $S = \{A_1, A_2, \dots, A_i, \dots, A_N\}$ ;

Given  $\mu_W, \sigma_W, \mu_{A_i}, \sigma_{A_i}$  where  $i \in \{1 \dots N\}$ ; and  $\rho_{W,i}, \rho_{i,j}$ , where  $i, j \in \{1 \dots N\}$ ;

To choose  $\Omega^* \in 2^S$ , such that  $W + \Omega^* \succ_u W + \Omega, \forall \Omega \in 2^S$

It can also be modeled in mathematical language. Recall that

- $\delta_i \in \{0,1\}, i=1, \dots, N$ : indicator of whether investment opportunity  $i$  should be chosen
- $r$ : Arrow-Pratt Coefficient of Absolute Risk Aversion
- $\rho_{ij}$ : correlation between two investment opportunities  $i$  and  $j$
- $\mu_i$ : expected return of investment opportunity  $i$
- $\sigma_i$ : standard deviation of investment opportunity  $i$

The problem can be formulated as

$$\underset{\{\delta_i\}_{i=1\dots N}}{MAX} \quad r \cdot \sum_{i=1}^N \mu_i \cdot \delta_i - \frac{r^2}{2} \cdot \sum_{i=1}^N \sum_{j=i}^N \rho_{ij} \sigma_i \sigma_j \delta_i \delta_j \quad \dots\dots (4.1)$$

$$such\ that \ \delta_i \in \{0,1\}, i = 1, \dots, N$$

As noticed, this is a Integer Quadratic Programming (IQP) problem.

Example 4.1:

Zeus Ltd. is an Asia-based private equity firm with its main business focus on Fund of Funds investment.

With over \$1 billion assets under management, Zeus Ltd. developed an extremely disciplined and independently recognized investment method.

In the first place, the company carefully studied its investors' risk preference which has the characteristic of Constant Absolute Risk Aversion (or "CARA") with  $r=1$ .

Hence,

$$U(x) = a - be^{-rx} = a - be^{-x}$$

$$E[u(X)] = a - b \cdot e^{r\mu_x - \frac{r^2\sigma_x^2}{2}} = a - be^{\mu_x - \frac{\sigma_x^2}{2}}$$

In addition, Zeus Ltd. made an extensive market research and identified 60 potential investment opportunities, characterized by

- Expected return:  $\mu_i$  with  $i = 1 \dots 60$
- Standard deviation):  $\sigma_i$  with  $i = 1 \dots 60$
- Correlation:  $\rho_{ij}$  with  $i, j = 1 \dots 60$



Detailed data of  $\mu_i$ ,  $\sigma_i$  and  $\rho_{ij}$  are given in the tables in *Appendix A*.

With the above information, Zeus Ltd. needs to make investment decisions in the investors' best interest, i.e. to maximize the expected utility function.

#### 4.2 Exact Approach – Exhaustive Search Algorithm

Start with the Exhaustive Search Algorithm (*Appendix C*), which is an exact approach to find the global optimum.

Given the initial wealth, this algorithm tries out all the possible subsets of  $S = \{A_1, A_2, \dots, A_i, \dots, A_N\}$  and identifies the best one.

Algorithm 4.1 (Exhaustive Search Algorithm):

```

initialize  $W$  ;

for all  $\Omega \subseteq S$  {

    if ( $W + \Omega > W$ ) {

         $\Omega^* = \Omega$  ;  $W = W + \Omega$  ;

    }

}

END

```

Although this algorithm gives the best global optimum, it is at the price of very expensive time efficiency.

Let  $T(n)$  be the solving time, depending on the problem size  $n$ . In the case of Exhaustive Search Algorithm,  $T(n) = O\left(2^{|A_1, A_2, \dots, A_i, \dots, A_N|}\right) = O(e^{|\{A_i\}|})$ .

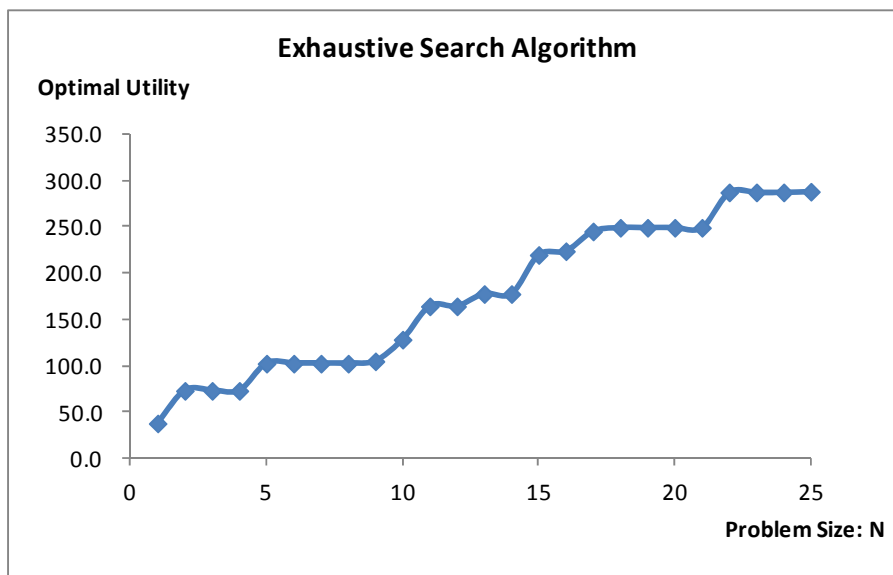
### **Case Study**

Apply Exhaustive Search Algorithm in the problem of Zeus Ltd.

The optimal utility result is always found (*Figure 4.1*), which makes sense to be a non-decreasing function because it is always preferred to have more investment choices.

However, the solving time <sup>2</sup> (*Figure 4.2*) increases very fast at an exponential rate.

For example, at  $N = 25$ , it takes  $t = 11,757$  seconds (almost 3 hours) to find the global optimum.



*Figure 4.1 - Exhaustive Search Algorithm - Optimal Utility (Problem Size: 1 – 25)*

<sup>2</sup> Computer: Hewlett-Packard; Model: HP Pavilion dm1 Notebook PC; Operating System: Windows 7 Home Premium Service Pack 1; Processor: AMD E-350 Processor 1.60 GHz; Installed memory (RAM): 4.00GB (3.60GB usable); System type: 64-bit Operating System

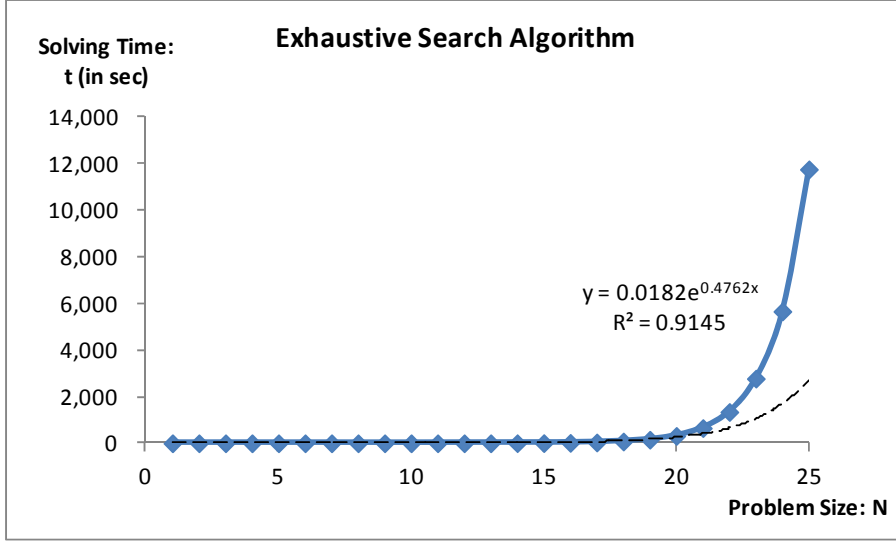


Figure 4.2 - Exhaustive Search Algorithm - Solving Time (Problem Size: 1 – 25)

Solving Time Regression:  $t = 0.0182 \cdot e^{0.4762 N}$  with  $R^2 = 0.9145$

### 4.3 Exact Approach – CPLEX Optimization

IBM ILOG CPLEX Optimization Studio (or “CPLEX”)<sup>3</sup> is an optimization software package for LP (Linear Programming), MILP (Mixed Integer Linear Programming), MIQP (Mixed Integer Quadratic Programming) etc.

Our problem can be formulated in the language of CPLEX as below:

$$\underset{\{\delta_i\}_{i=1 \dots N}}{\text{MAX}} \quad \frac{1}{2} \cdot X' \cdot H \cdot X + f \cdot X \quad \dots\dots (4.2)$$

$$\text{Where } f = [r \cdot \mu_1 \quad \dots \quad r \cdot \mu_i \quad \dots \quad r \cdot \mu_N]$$

$$X = [\delta_1 \quad \dots \quad \delta_i \quad \dots \quad \delta_N]'$$

$$H = \begin{bmatrix} \rho_{1,1} \sigma_1 \sigma_1 & \dots & \rho_{1,N} \sigma_1 \sigma_N \\ \dots & \rho_{i,j} \sigma_i \sigma_j & \dots \\ \rho_{N,1} \sigma_N \sigma_1 & \dots & \rho_{N,N} \sigma_N \sigma_N \end{bmatrix}$$

<sup>3</sup> IBM ILOG CPLEX Optimization Studio Academic Research Edition is used in this paper  
<http://www-01.ibm.com/software/integration/optimization/cplex-optimization-studio/>

The detailed program coding (under Matlab<sup>4</sup>) can be found in *Appendix D*.

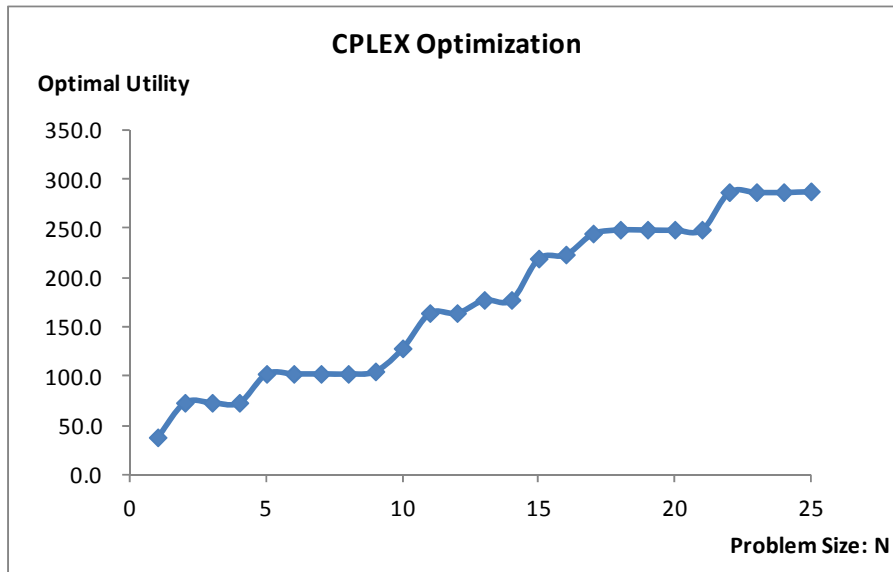
It can be shown that although CPLEX calculates faster than Exhaustive Search Algorithm, the running time still increases exponentially when problem size gets large.

### **Case Study**

Apply CPLEX Optimization in the problem of Zeus Ltd.

It is noticed that CPLEX dominates Exhaustive Search Algorithm: both find the same global optimum results (*Figure 4.3*), while CPLEX always runs faster than Exhaustive Search (*Figure 4.4*).

For example, at problem size of  $N = 25$ , CPLEX only requires  $t = 59.4$  seconds (vs. 3 hours by Exhaustive Search Algorithm) to find the global optimum.



*Figure 4.3 - CPLEX - Optimal Utility (Problem Size: 1 – 25)*

<sup>4</sup> Matlab 7.12.0.635 (R2011a) 64-bit (win64) is used in this paper

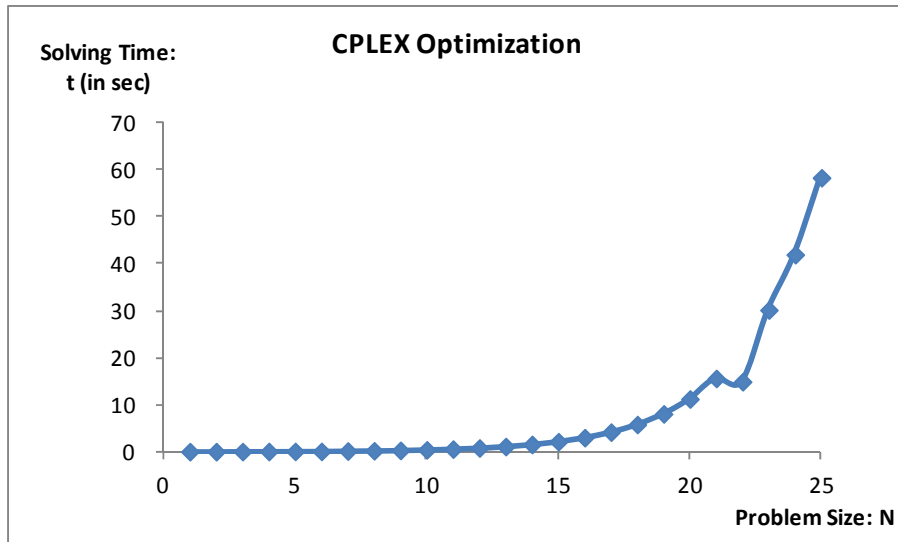


Figure 4.4 - CPLEX - Solving Time (Problem Size: 1 – 25)

However, when problem size increases, CPLEX's solving time also follows exponential growth trend (Figure 4.6). At  $N = 60$ , it takes CPLEX  $t = 5,972,118$  seconds (almost 2 months) to find the global optimum.

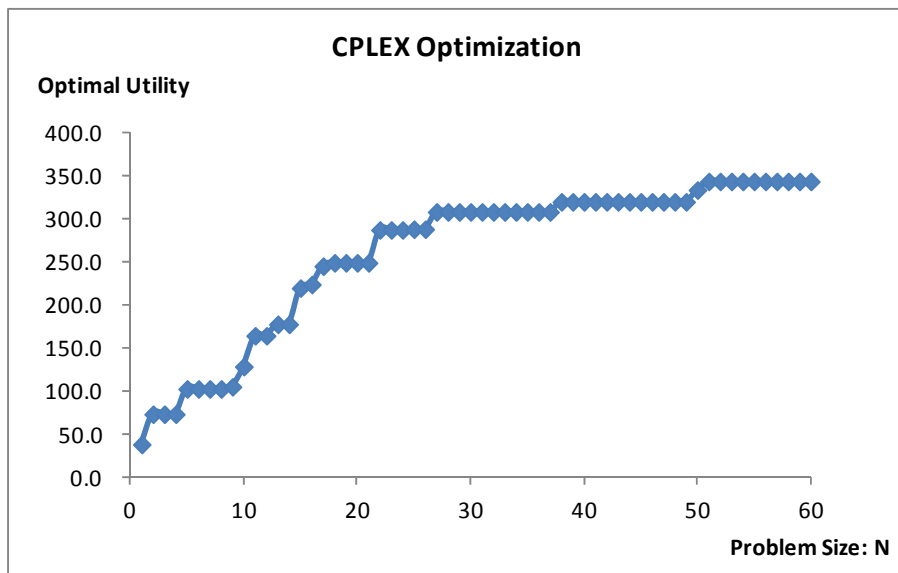


Figure 4.5 - CPLEX - Optimal Utility (Problem Size: 1 – 60)

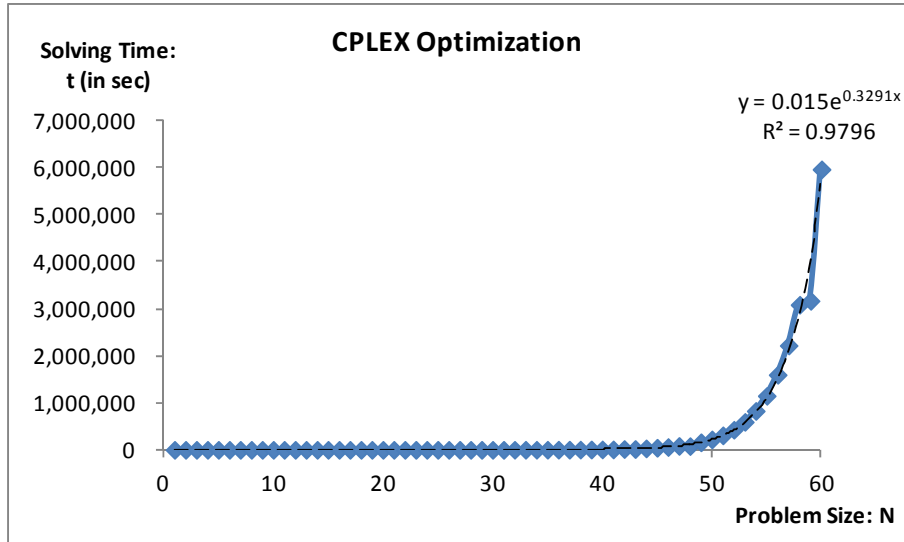


Figure 4.6 - CPLEX - Solving Time (Problem Size: 1 – 60)

Solving Time Regression:  $t = 0.015 \cdot e^{0.3291 N}$  with  $R^2 = 0.9796$

#### 4.4 Heuristic Approach – Greedy Algorithm

A Greedy Algorithm is the most intuitive methodology to try out all the investment candidates one at a time, and the utility result will keep improving after going through the entire set  $S = \{A_1, A_2, \dots, A_i, \dots, A_N\}$

Algorithm 4.2 (Greedy Algorithm):

initialize  $W$  and  $\Omega = \Phi$ ;

for ( $i = 1$  to  $|S|$ ) {

    if ( $W + A_i > W$ ) {

$W = W + A_i$ ;  $\Omega = \Omega \cup \{A_i\}$

    }

}

END

This algorithm is simple, intuitive and time efficient with linear complexity. Recall that  $T(n)$  is the solving time; and in the case of Greedy Algorithm,  $T(n) = O(|\{A_i\}|)$ .

However, as what can be seen later, the Greedy Algorithm generally does not guarantee a global optimum.

#### 1) Zero Correlation

It can be shown that if there is zero correlation among the initial portfolio and investment opportunity candidates, the Greedy Algorithm gives the optimal solution.

#### Theorem 4.1:

With the property of CARA utility function and normal distribution random variables, if there is no correlation among the initial portfolio and any investment opportunities, the Greedy Algorithm will produce the global optimal solution.

#### Proof:

The iso-utility curve has the form of  $r\mu_x - \frac{r^2\sigma_x^2}{2} = c$ .

Let  $\mu_x$  and  $\sigma_x^2$  be for the y and x axes respectively, and so the function becomes a straight line with gradient  $\frac{r}{2}$ .

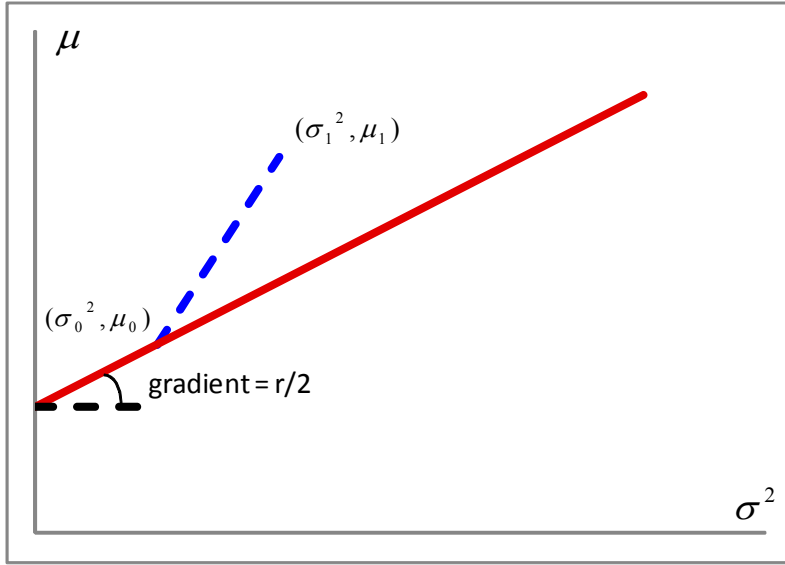


Figure 4.7 - Illustration of Greedy Algorithm

$(\sigma_1^2, \mu_1)$  is preferred over  $(\sigma_0^2, \mu_0)$ , if and only if  $(\sigma_1^2, \mu_1)$  lies above the straight line passing through  $(\sigma_0^2, \mu_0)$ . Equivalently, the vector connecting  $(\sigma_0^2, \mu_0)$  and  $(\sigma_1^2, \mu_1)$  must have a gradient greater than  $\frac{r}{2}$ .

The gradient of vector  $(\sigma_1^2 - \sigma_0^2, \mu_1 - \mu_0)$  is given by  $\frac{\mu_1 - \mu_0}{\sigma_1^2 - \sigma_0^2}$ .

If  $W$  and  $A$  are uncorrelated,  $\mu_{W+A} = \mu_W + \mu_A$  and  $\sigma_{W+A}^2 = \sigma_W^2 + \sigma_A^2$

$$\Rightarrow \frac{\mu_{W+A} - \mu_W}{\sigma_{W+A}^2 - \sigma_W^2} = \frac{\mu_A}{\sigma_A^2} \text{ which is completely independent of the initial wealth.}$$

Hence, any investment opportunity  $A_i$  with  $\frac{\mu_{A_i}}{\sigma_{A_i}^2} > \frac{r}{2}$  should be and will be included

in the global optimal solution, by using the Greedy Algorithm.

Q.E.D



## 2) Non-zero Correlation

While the Greedy Algorithm works well for a portfolio without any correlation, it fails in the case of a portfolio of random variables correlated with each other.

Below is an example where  $A$  is a “preferred” candidate by the Greedy Algorithm, but is not in the best decision portfolio. In other words,

$W + A \succ_u W + B_i, \forall B_i \in S$ ; however,

$A_i \notin \Omega$  where  $W + \Omega \succ_u W + \Psi_i, \forall \Psi_i \subseteq S$ ;

### Example 4.2:

Assume an investor’s risk profile is characterized by Arrow-Pratt Coefficient  $r = 2$ . With an initial wealth  $W$ , he is given six investment opportunities: one opportunity  $A$  and five identical opportunities  $B$ . He would like to choose a subset of the six investment opportunities that maximizes his expected utility.

$A$ ,  $B$  and  $W$  are characterized in the tables below:

Table 4.1 - Example of Greedy Algorithm (Mean and Standard Deviation)

Portfolio	$\sigma$	$\mu$
W	0.5	20
A	2.5	10
B	0.5	3
2*B	1.0	6
3*B	1.5	9
4*B	2.0	12
5*B	2.5	15
B+A	2.6	13
2*B+A	2.9	16
3*B+A	3.1	19
4*B+A	3.5	22
5*B+A	3.8	25

Table 4.2 - Example of Greedy Algorithm (Correlation)

	W	A	B
W	1	0.35	0
A	0.35	1	0.35
B	0	0.35	1

The mean, standard deviation and expected utility of different possible combinations are summarized in *Table 4.3* below:

*Table 4.3 - Example of Greedy Algorithm (all portfolio combinations)*

Portfolio	$\sigma$	$\mu$	$r\mu - (r^2\sigma^2)/2$
W	0.5	20	39.5
W+A	2.7	30	45.3
W+B	0.7	23	45.0
W+2*B	1.1	26	49.5
W+3*B	1.6	29	53.0
W+4*B	2.1	32	55.5
W+5*B	2.5	35	57.0
W+B+A	2.9	33	49.0
W+2*B+A	3.2	36	51.8
W+3*B+A	3.5	39	53.5
W+4*B+A	3.9	42	54.3
W+5*B+A	4.2	45	54.0

*Table 4.4 - Example of Greedy Algorithm (combinations sorted by utility)*

Rank	Portfolio	$r\mu - (r^2\sigma^2)/2$
1	W+5*B	57.0
2	W+4*B	55.5
3	W+4*B+A	54.3
4	W+5*B+A	54.0
5	W+3*B+A	53.5
6	W+3*B	53.0
7	W+2*B+A	51.8
8	W+2*B	49.5
9	W+B+A	49.0
10	W+A	45.3
11	W+B	45.0
8	W+2*B	49.5

Given the initial wealth  $W$ , individually opportunity  $A$  is better than  $B$  (i.e.  $W + A > W + B$ ). Therefore, the first step of Greedy Algorithm will choose  $A$  to be included the portfolio.

However, globally the best portfolio is  $W + 5B$ , without the presence of  $A$  (i.e.  $W + 5B + A < W + 5B$ ).

Graphically, It is obvious that opportunity  $A$  can improve the expected utility at point  $P$  better than  $B$  can do (i.e.  $W + A > W + B$ ); however, it has an adverse effect on utility at point  $Q$  (i.e.  $W + 5B + A < W + 5B$ ), which is the global optimal solution.

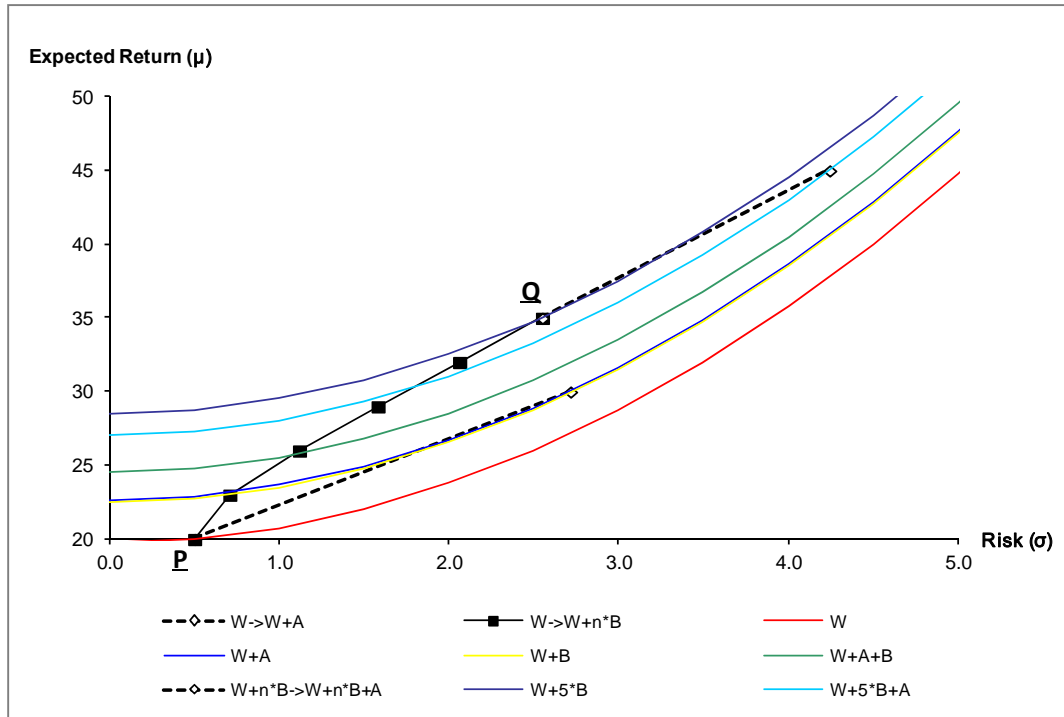


Figure 4.8 – An Example of the Failure of Greedy Algorithm

If the Greedy Algorithm is applied to the example above, the evolution of the decision portfolio will be  $\{A\} \rightarrow \{A, B\} \rightarrow \dots \rightarrow \{A, B, B, B, B\}$ , and the final solution is not the optimal solution, which should be  $\{B, B, B, B, B\}$

### Case Study

Apply the Greedy Algorithm to the problem of Zeus Ltd.

There is a significant improvement on the solving time (Figure 4.10); the problem with  $N = 60$  requires less than 1 second (vs. 2 months by CPLEX at  $N = 60$ ).

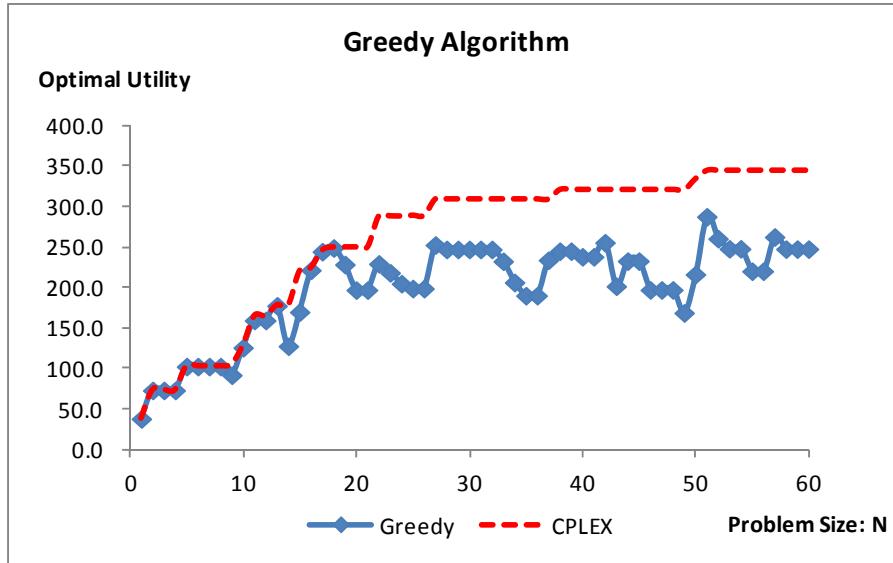


Figure 4.9 - Greedy Algorithm - Optimal Utility (Problem Size: 1 – 60)

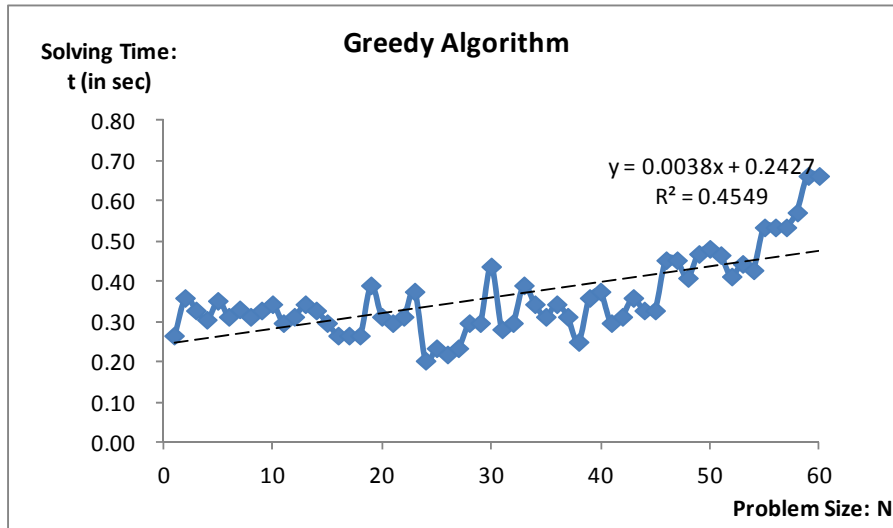


Figure 4.10 - Greedy Algorithm - Solving Time (Problem Size: 1 – 60)

Solving Time Regression:  $t = 0.0038N + 0.2427$  with  $R^2 = 0.4549$

Note that  $R^2$  is small. This is due to the limited sample size (problem size 1 to 60). If applying the same Greedy Algorithm to problem with size 1 to 240,  $R^2$  will be improved to above 0.7.

However, most of the optimal utilities found are local solutions (Figure 4.9), which are on average approximately two thirds of the global optimums.

#### 4.5 Heuristic Approach – Hill Climbing Algorithm

The problem of the Greedy Algorithm is that once an investment opportunity is taken, it always remains in the decision portfolio, even if the utility can be improved by taking this opportunity out of the portfolio. In other words, this methodology takes only one investment opportunity at a time and never tries to consider two (or more) candidates together.

Inspired by the famous travelling salesman problem (Johnson & McGeoch, 1995), the “k-opt Hill Climbing Algorithm” is proposed in this thesis, where  $k$  is the number of candidates to consider at each iteration.

In the case of k-opt, each iteration tries to improve the expected utility by adding in  $m$  new candidates ( $m$  in) and removing  $n$  existing investment opportunities ( $n$  out), as long as  $m + n \leq k$ .

Algorithm 4.2 (k-opt Hill Climbing Algorithm):

initialize  $W$  and  $\Omega = \Phi$ ;

find  $W$  and  $\Omega$  using 1-opt ... (k-1)-opt Hill Climbing Algorithm

for ( $i' = 1$  to  $|S|$ ) {

    for ( $i'' = i'$  to  $|S|$ ) {

        ...

    for ( $i^{(k)} = i^{(k-1)}$  to  $|S|$ ) {

$W_0 = W$  ;

```

for all  $n \in \{1...k\}$  {

    if  $A_{i(n)} \notin \Omega$ ;  $W = W + A_{i(n)}$ ;  $\Omega = \Omega \cup \{A_{i(n)}\}$ 

    if  $A_{i(n)} \in \Omega$ ;  $W = W - A_{i(n)}$ ;  $\Omega = \Omega \setminus \{A_{i(n)}\}$ 

}

if  $W < W_0$ ; {

     $W = W_0$ 

}

}

}

}

END

```

The table below summarizes the possibilities of the “k-opt Hill Climbing Algorithm” when  $k = 1...4$ , and it can be noticed that the Greedy Algorithm is a special case of the “k-opt Hill Climbing Algorithm” when  $k = 1$

Table 4.5 - Example of  $k$ -opt Hill Climbing Algorithm ( $k=1,2,3,4$ )

Hill Climbing Algorithm			
Greedy / 1-opt	2-opt	3-opt	4-opt
1 in 0 out	<b>1 in 0 out</b>	<b>1 in 0 out</b>	<b>1 in 0 out</b>
0 in 1 out	<b>0 in 1 out</b>	<b>0 in 1 out</b>	<b>0 in 1 out</b>
	1 in 1 out	<b>1 in 1 out</b>	<b>1 in 1 out</b>
	2 in 0 out	<b>2 in 0 out</b>	<b>2 in 0 out</b>
	0 in 2 out	<b>0 in 2 out</b>	<b>0 in 2 out</b>
		3 in 0 out	<b>3 in 0 out</b>
		2 in 1 out	<b>2 in 1 out</b>
		1 in 2 out	<b>1 in 2 out</b>
		0 in 3 out	<b>0 in 3 out</b>
			4 in 0 out
			3 in 1 out
			2 in 2 out
			1 in 3 out
			0 in 4 out
$O( \{A_i\} )$	$O( \{A_i\} ^2)$	$O( \{A_i\} ^3)$	$O( \{A_i\} ^4)$

This algorithm is designed to have the property that

$$\{k\text{-opt}\} \subseteq \{(k+1)\text{-opt}\}$$

Theorem 4.2:

The solution of k-opt is always suboptimal to (k+1)-opt.

Proof:

k-opt algorithm is defined to try out all the possibilities “m in, n out” where  $m + n \leq k$

$$\{m \in \mathbb{N}, n \in \mathbb{N} \mid m + n \leq k\} \subseteq \{m \in \mathbb{N}, n \in \mathbb{N} \mid m + n \leq k + 1\}$$

$\Rightarrow$  (k+1)-opt tries out all the possibilities that k-opt will go through.

Q.E.D

In terms of Complexity

$$T(n)_{k-opt} = O(|\{A_i\}| \cdot (|\{A_i\}| - 1) \cdot \dots \cdot (|\{A_i\}| - (k - 1))) + T(n)_{(k-1)-opt} = O(|\{A_i\}|^k)$$

By compromising on time efficiency, k-opt Hill Climbing Algorithm produces a better suboptimal solution than Greedy Algorithm (by *Theorem 4.2*); however, it still does not guarantee the global optimum, **unless**  $k = |\{A_i\}|$ , in which case the complexity is

$$T(n)_{|\{A_i\}|-opt} = O(|\{A_i\}|^{|\{A_i\}|}), \text{ even worse than Exhaustive Search Algorithm.}$$

1) Zero Correlation

Theorem 4.3:

With the property of CARA utility function and normal distribution random variables, if there is no correlation among the initial portfolio and any investment opportunities, the “k-opt Hill Climbing Algorithm” will produce the global optimal solution.



Proof:

By *Theorem 4.1*,  $\{\text{global optimum}\} = \{\text{Greedy Algorithm solution}\}$

By *Theorem 4.2*,  $\{\text{Greedy Algorithm solution}\} = \{1\text{-opt solution}\}$  which is always suboptimal to  $\{k\text{-opt solution} \mid k > 1\}$

$\Rightarrow \{k\text{-opt Hill Climbing Algorithm solution}\} = \{\text{global optimum}\}.$

Q.E.D

## 2) Non-zero Correlation

It can be shown that  $(k+1)$ -opt is an improved algorithm of  $k$ -opt; in particular, 2-opt Hill Climbing could produce a better solution than Greedy Algorithm (1-opt).

Taking again *Example 4.2*, the decision making order will be  $\{A\} \rightarrow \{A, B\} \rightarrow \dots \rightarrow \{A, B, B, B, B\} \rightarrow \{B, B, B, B\} \rightarrow \{B, B, B, B, B\}$ , which is the global optimal solution.

However, global optimality is not guaranteed and below is another example where 2-opt Hill Climbing fails to find out the global optimum.

### Example 4.3

Initial wealth and investment opportunities are characterized as in the tables:

*Table 4.6 - Example of Hill Climbing Algorithm (Mean and Standard Deviation)*

	$\sigma$	$\mu$
W	2.0	10.0
A	2.0	0.5
B	2.0	1.4
C	2.1	4.8
D	2.2	4.6

Table 4.7 - Example of Hill Climbing Algorithm (Correlation)

	<i>W</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>
<i>W</i>	1.0	-0.9	-0.3	-0.3	-0.3
<i>A</i>	-0.9	1.0	-0.1	0.3	0.3
<i>B</i>	-0.3	-0.1	1.0	0.2	0.3
<i>C</i>	-0.3	-0.1	0.2	1.0	0.0
<i>D</i>	-0.3	-0.3	0.3	0.0	1.0

Table 4.8 - Example of Hill Climbing Algorithm (combinations sorted by utility)

Rank	$r\mu-(r^2\sigma^2)/2$	Portfolio
1	22.4	W+C+D
2	21.4	W+A+B
3	20.1	W+A+C
4	19.8	W+A+C+D
5	19.5	W+A+B+C
6	19.4	W+A+D
7	19.3	W+A
8	18.0	W+C
9	17.4	W+D
10	16.7	W+A+B+D
11	14.9	W+B+C
12	14.7	W+B+C+D
13	14.5	W+A+B+C+D
14	12.3	W+B+D
15	12.2	W
16	12.1	W+B

Table 4.9 - Example of Hill Climbing Algorithm (all portfolio combinations)

Portfolio	$\sigma$	$\mu$	$r\mu-(r^2\sigma^2)/2$
W	1.97	10.0	12.2
W+A	0.92	10.5	19.3
W+B	2.32	11.4	12.1
W+C	2.40	14.8	18.0
W+D	2.42	14.6	17.4
W+A+B	1.08	11.9	21.4
W+A+C	2.28	15.2	20.1
W+A+D	2.31	15.0	19.4
W+B+C	2.96	16.2	14.9
W+B+D	3.13	16.0	12.3
W+C+D	2.85	19.3	22.4
W+A+B+C	2.63	16.7	19.5
W+A+B+D	2.84	16.4	16.7
W+A+C+D	3.14	19.8	19.8
W+B+C+D	3.66	20.7	14.7
W+A+B+C+D	3.74	21.2	14.5

It can be verified that the 2-opt Hill Climbing Algorithm will construct the decision

portfolio in the order  $\{A\} \rightarrow \{A, B\}$ , even though  $\{A, B\} \prec_u \{C, D\}$

As a matter of fact, both  $\{A, B\}$  and  $\{C, D\}$  are local optimums, and  $\{C, D\}$  is also the global optimum. Since the algorithm starts with  $A$ , it falls into the trap of the local solution  $\{A, B\}$ .

### **Case Study**

Apply the “k-opt Hill Climbing Algorithm” in the problem of Zeus Ltd.

As far as solving time is concerned, the 2-opt Hill Climbing Algorithm (*Figure 4.12*) follows a polynomial trend with a power of two (quadratic). It is not as fast as the Greedy Algorithm (i.e. 1-opt Hill Climbing); however, there is a significant improvement of the search results (*Figure 4.11*) with most of them (above 90%) being the global optimums.

The 3-opt Hill Climbing Algorithm further improved the global optimal results ratio (*Figure 4.13*) to 98%. However, this is at the price of time efficiency (*Figure 4.14*), which is a polynomial function with a power of three. At  $N = 60$ , the 3-opt Hill Climbing requires  $t_{3-opt} = 55.2$  seconds, vs.  $t_{2-opt} = 4.7$  seconds by 2-opt.

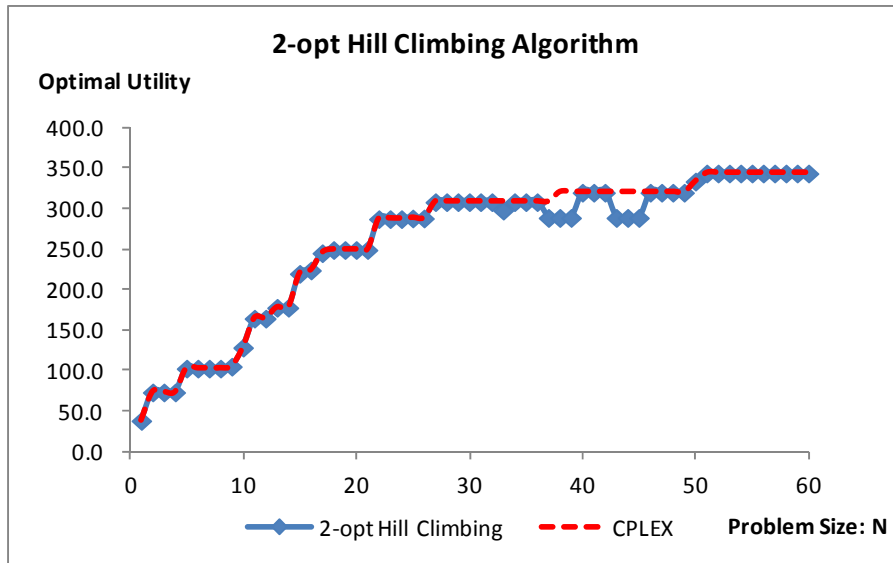


Figure 4.11 - 2-opt Hill Climbing Algorithm - Optimal Utility (Problem Size: 1 – 60)

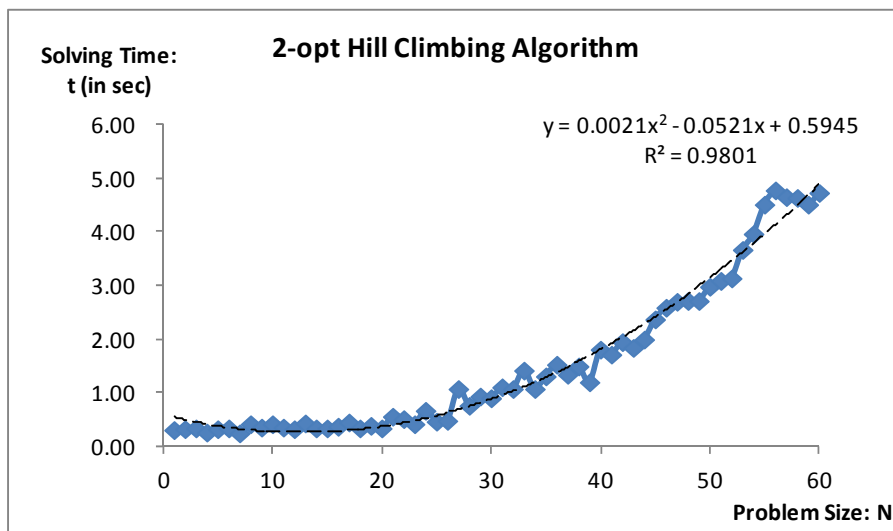


Figure 4.12 - 2-opt Hill Climbing Algorithm - Solving Time (Problem Size: 1 – 60)

Solving Time Regression (2-opt):  $t = 0.0021 N^2 - 0.0521 N + 0.5945$  with  $R^2 = 0.9801$

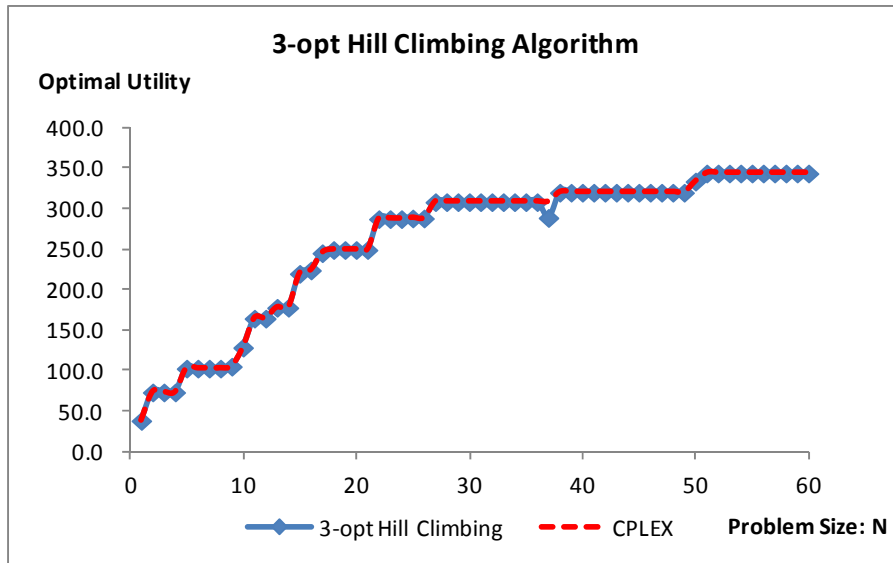


Figure 4.13 - 3-opt Hill Climbing Algorithm - Optimal Utility (Problem Size: 1 – 60)

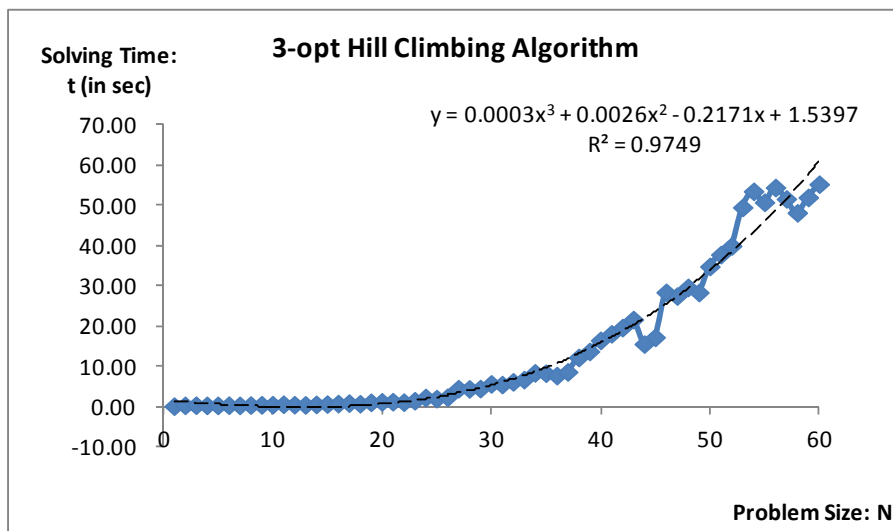


Figure 4.14 - 3-opt Hill Climbing Algorithm - Solving Time (Problem Size: 1 – 60)

Solving Time Regression (3-opt):  $t = 0.0003 N^3 + 0.0026 N^2 - 0.2171 N + 1.5397$

with  $R^2 = 0.9749$

#### 4.6 Heuristic Approach (Stochastic) – Random Restart Hill Climbing

In this thesis, two stochastic processes are proposed which do not guarantee the global optimal solution, but stand a chance to jump out of the traps of the local optimal solutions and eventually reach the global optimum.

The first one is the Random Restart Hill Climbing Algorithm. It is very similar to the previous Hill Climbing Algorithm, except that this algorithm repeats the same climbing process several times from different randomized starting points, each time arriving at a local optimal point.

Define  $M$  as the “Iteration Number”.  $M$  iterations find  $M$  local optimal points (some are repeated), with the hope that one of the local optimal solutions is the global optimum.

There is a trade-off between the execution time and the probability of finding the global optimal solution; this trade-off is controlled by the iteration number  $M$ .

Algorithm 4.4 (Random Restart  $k$ -opt Hill Climbing Algorithm):

```
for (m = 1 to M) {  
    randomize  $\Omega$  ;  
  
    for all  $A_i \in \Omega$  ;  $W = W + A_i$   
  
    k-opt Hill Climbing Algorithm with initial  $W$  and  $\Omega$   
  
} END
```

Reconsider *Example 4.3*. With a certain probability, the hill climbing process can start with either  $\{C\}$  or  $\{D\}$ , which eventually leads us to the global optimal point.

## Case Study

Apply the Random Restart k-opt Hill Climbing Algorithm to the problem of Zeus Ltd.

With  $M = 5$ , the solving time of Random 2-opt (Figure 4.16) is approximately 5 times the one by Simple 2-opt (still quadratic). However, the search result (Figure 4.15) has been improved to 100% global optimums for size  $N = 1 \dots 60$ .

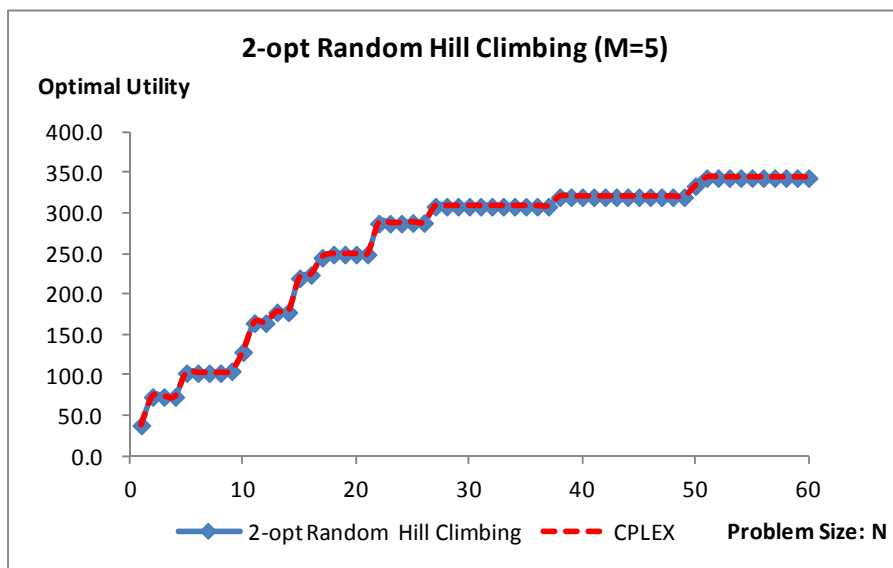


Figure 4.15 - 2-opt Random Hill Climbing Algorithm - Optimal Utility (Size: 1 – 60)

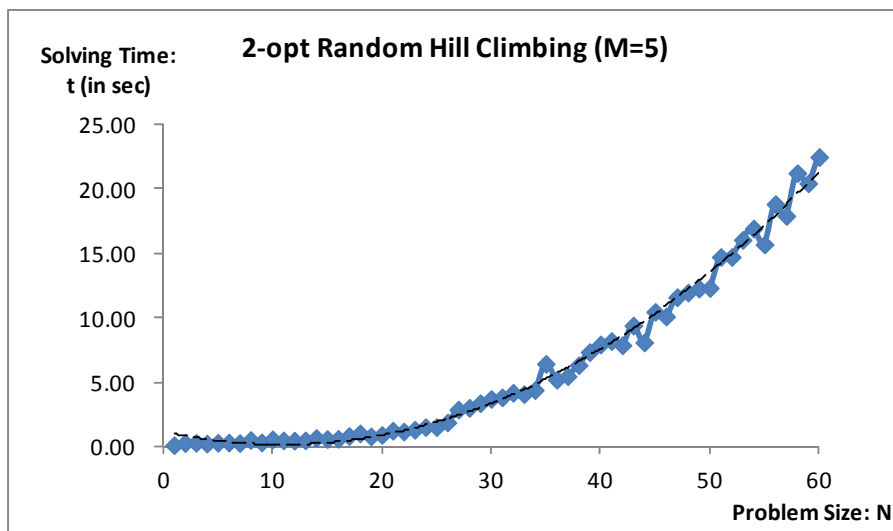


Figure 4.16 - 2-opt Random Hill Climbing Algorithm - Solving Time (Size: 1 – 60)

Solving Time Regression (Random 2-opt):

$$t_{2-opt (Random)}(M) \approx M \cdot t_{2-opt (Simple)} = M \cdot (0.0021 N^2 - 0.0521 N + 0.5945)$$

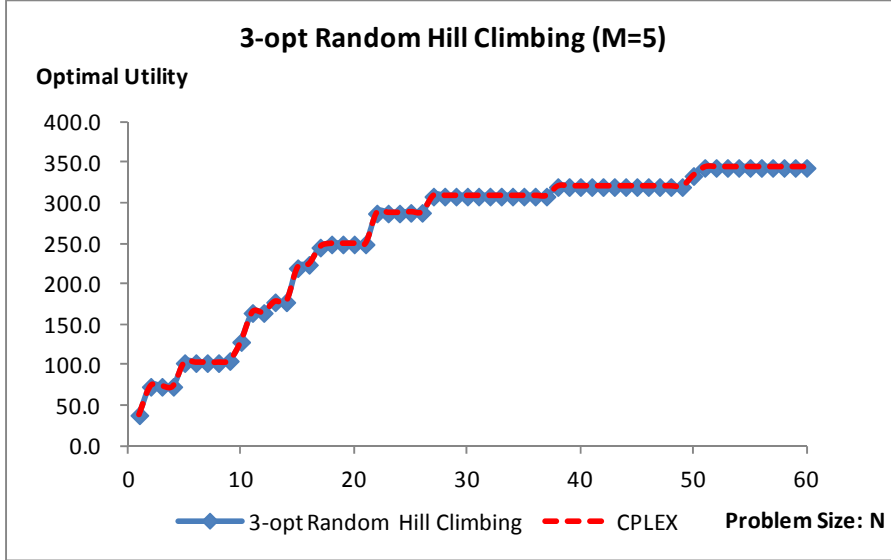


Figure 4.17 - 3-opt Random Hill Climbing Algorithm - Optimal Utility (Size: 1 – 60)

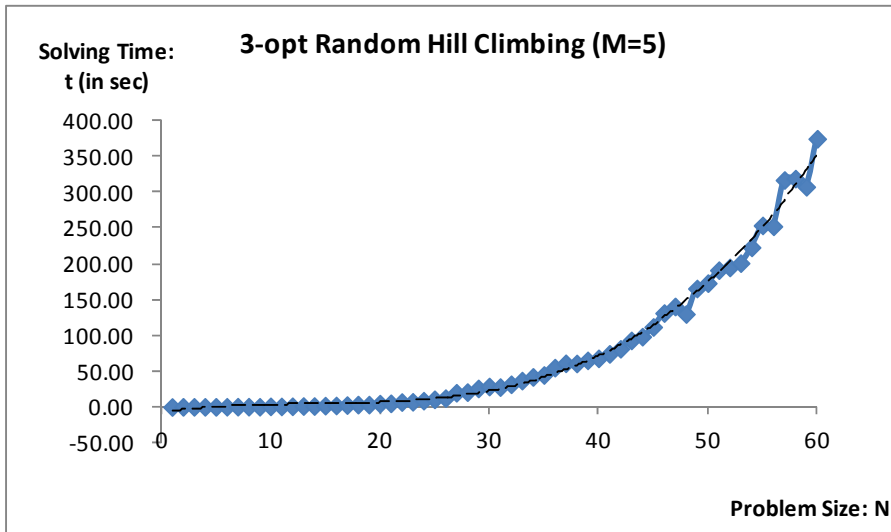


Figure 4.18 - 3-opt Random Hill Climbing Algorithm - Solving Time (Size: 1 – 60)

Solving Time Regression (Random 3-opt):

$$t_{3-opt (Random)}(M) \approx M \cdot t_{3-opt (Simple)} = M \cdot (0.0003 N^3 + 0.0026 N^2 - 0.2171 N + 1.5397)$$



#### 4.7 Heuristic Approach (Stochastic) – Stochastic Gradient Ascent

Lastly, the Stochastic Gradient Ascent Algorithm is introduced in this chapter.

Compared to the previous approaches, which only focus on moving in a direction of better expected utility value, the Stochastic Gradient Ascent Algorithm spares a part of its attention to make an effort to jump from a local optimal point to another local optimal point, even at the cost of decreasing the utility value.

A fraction  $\alpha \in (0,1)$  is used here. With probability  $1 - \alpha$ , the searching process is the same as the previously studied k-opt Hill Climbing Algorithm; at the same time, with probability  $\alpha$ , the searching direction is completely random, and hence stands a chance to fall into the adjacent local optimal point.

Similar to the Random Restart Hill Climbing Algorithm, the Stochastic Gradient Ascent Algorithm has a trade-off between execution time and probability to jump out of the trap of the local optimal solutions and to find the global optimal solution. This trade-off is controlled by the probability factor  $\alpha$ .

Algorithm 4.5 (k-opt Stochastic Gradient Ascent Algorithm):

initialize  $W$  and  $\Omega = \Phi$ ;

while (utility can be improved by either random gradient OR k-opt Hill Climbing) {

    with probability  $\alpha$

        Search with random gradient

    with probability  $1 - \alpha$

```

Search with k-opt Hill Climbing Algorithm

}

END

```

Note again that a global optimality is not guaranteed; furthermore, it is possible not to be aware that one of the local optimal solutions is the global optimal solution, even if it has been found during the searching process.

### **Case Study**

Apply the “k-opt Stochastic Gradient Ascent Algorithm” in the problem of Zeus Ltd.

With  $\alpha = 10\%$ , the solving time of 2-opt follows a quadratic trend (*Figure 4.20*); however, the optimal utility result is very volatile (~75% accuracy): either at global optimum or at a huge gap from the global optimal solution (*Figure 4.19*).

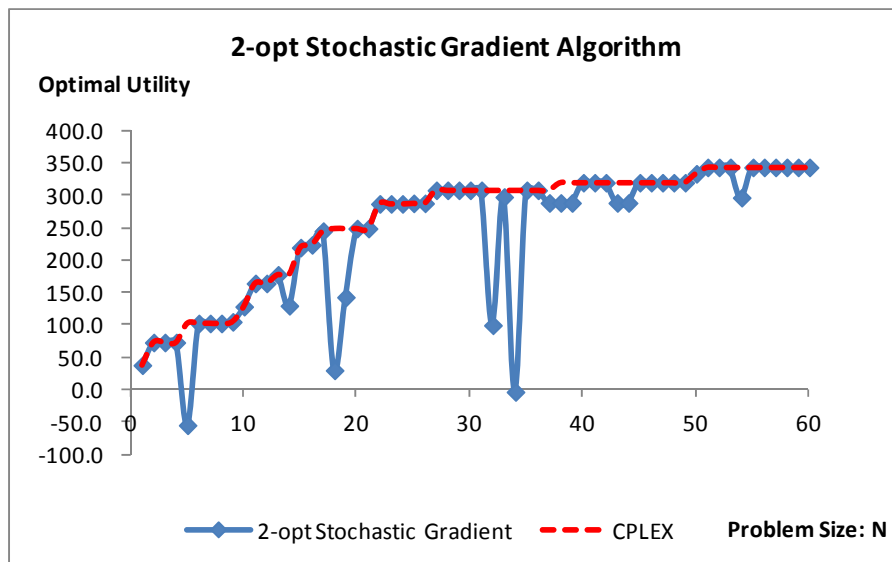


Figure 4.19 - 2-opt Stochastic Gradient Ascent Algorithm - Optimal Utility (Size: 1 – 60)

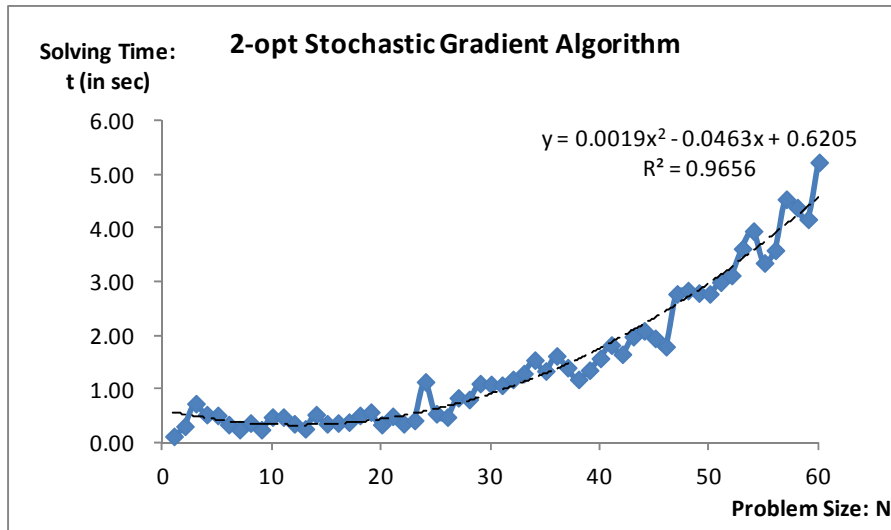


Figure 4.20 - 2-opt Stochastic Gradient Ascent Algorithm - Solving Time (Size: 1 – 60)

Solving Time Regression (2-opt):  $t = 0.0019N^2 - 0.0463N + 0.6205$  with  $R^2 = 0.9656$

#### 4.8 Comparison among Algorithms

For the risk-adjusted multiple investment decision facing Zeus Ltd, different algorithms have been applied in an attempt to solve the problem. The “Optimal Utility Level” and “Solving Time” are summarized in the figures below (Figure 4.21 & Figure 4.22)

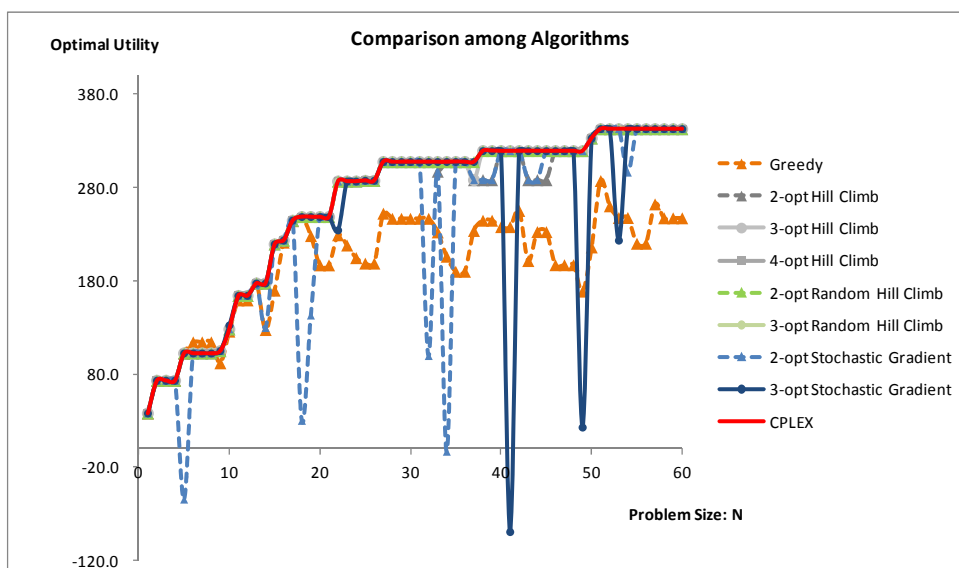


Figure 4.21 – Comparison among Algorithms - Optimal Utility (Problem Size: 1 – 60)

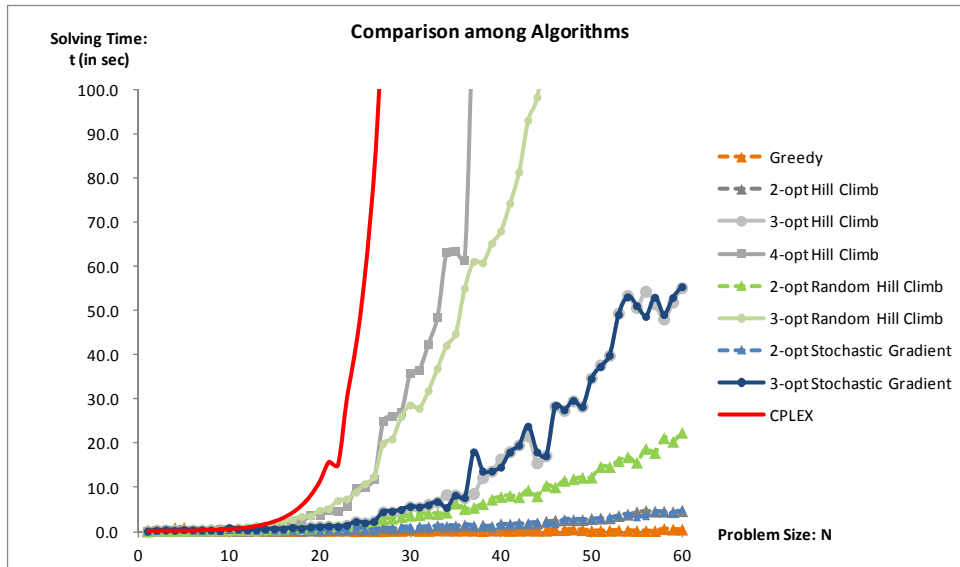


Figure 4.22 – Comparison among Algorithms - Solving Time (Problem Size: 1 – 60)





















As noticed, each algorithm has its own pros and cons.

In terms of **time efficiency**, Greedy Algorithm is the fastest, followed by 2-opt Hill Climbing, 2-opt Stochastic Gradient, 2-opt Random Hill Climbing, 3-opt Stochastic Gradient, 3-opt Random Hill Climbing and so on.

As far as **optimal level** is concerned, CPLEX, k-opt Random Hill Climbing and k-opt Simple Hill Climbing Algorithms find most of the global optimums, while Greedy Algorithm consistently underperforms and k-opt Stochastic Gradient displays very volatile results.

In conclusion, no single algorithm tops the list for both criteria; hence, a compromising balance is necessary depending on the user's needs. Below is a table (Table 4.10) qualitatively summarizing each algorithm's characteristics

Table 4.10 - Comparison among Algorithms (Time Efficiency vs. Optimal Level)

Algorithms	Time Efficiency <sup>5</sup>	Optimal Level <sup>6</sup>
Exhaustive Search		
CPLEX		
Greedy		
2-opt Hill Climbing		
3-opt Hill Climbing		
4-opt Hill Climbing		
2-opt Random Hill Climbing		
3-opt Random Hill Climbing		
2-opt Stochastic Gradient		
3-opt Stochastic Gradient		

One finding is that the highlighted two algorithms (Random Restart 2-opt Hill Climbing and Random Restart 3-opt Hill Climbing) seem to be the most outstanding approaches. A satisfactory optimal level is obtained, yet the solving time is relatively inexpensive.

In the next chapter, both algorithms will be our focus to solve risk-adjusted and time-adjusted problems.

<sup>5</sup> For “time efficiency”, **full round** means the most time efficient, better or equal to  $O(|\{A_i\}|^2)$ ; **a quarter round** means the least time efficient, requiring at least time  $O(e^{|\{A_i\}|})$

<sup>6</sup> For “optimal level”, **full round** means high possibility to find the global optimum; **a quarter round** means limited chance to identify the global solution

## Chapter 5 Risk-Adjusted and Time-Adjusted Multiple Investment Decisions

### 5.1 Problem Description

In *Chapter 4*, the strategies have been studied to make the best investments decision under investor's risk-adjusted utility behaviour. However, the time value of future return has not yet been taken into consideration.

In the framework of private equity investment, time adjustments include two aspects: on one side, the expected return of an investment opportunity is not constant with respect to time; on the other side, future cash flows (or "FV") should be discounted back to the present value ("PV") to make two investment yields comparable.

Hence,  $\mu_i(t)$  becomes a function of time. Recall that

- $i$ : the  $i^{\text{th}}$  investment opportunity
- $N$ : total number of available investment opportunities
- $\delta_i \in \{0,1\}, i=1,\dots,N$ : indicator of whether investment opportunity  $i$  should be chosen
- $t \geq 0$ : holding period of the entire portfolio
- $T_i \geq 0$ : maximum value holding period where the expected return of investment opportunity  $i$  can be improved to the maximum extent through the manager's operational value add (it can also be understood as the time period when the fund manager grows the company to a mature stage, and no more additional value can be created from the company)

- $a_i t^2 + b_i t + c_i$ : time-dependent expected return of investment opportunity  $i$  with  $a_i > 0$ ,  $b_i < 0$ ,  $-\frac{b_i}{2a_i} > 0$  and  $c_i \geq 0$ , so that the expected return is a decreasing function between  $[0, -\frac{b_i}{2a_i})$  and an increasing function between  $[-\frac{b_i}{2a_i}, +\infty)$ . The initial decreasing interval is due to some sunk cost such as transaction fees and due diligence expenses
- $\lambda_i(t)$ : indicator of whether portfolio holding period  $t$  is within or outside the maximum value holding period of investment opportunity  $i$ . And hence  $\lambda_{\{t \leq T_i\}} = 1$  if  $t \leq T_i$ ,  $\lambda_{\{t \leq T_i\}} = 0$  otherwise. Similarly  $\lambda_{\{t > T_i\}} = 1$  if  $t > T_i$ ,  $\lambda_{\{t > T_i\}} = 0$  otherwise.
- $d$ : discount rate of time value, or the risk free interest rate at which the amount will be compounded each period
- $r$ : Arrow-Pratt Coefficient of Absolute Risk Aversion
- $\rho_{ij}$ : correlation between two investment opportunities  $i$  and  $j$
- $\mu_i$ : expected return of investment opportunity  $i$
- $\sigma_i$ : standard deviation of investment opportunity  $i$

The problem should be re-written as

$$\underset{\{\delta_i\}_{i=1 \dots N}, t}{MAX} \quad r \cdot \sum_{i=1}^N \mu_i(t) \cdot \delta_i - \frac{r^2}{2} \cdot \sum_{i=1}^N \sum_{j=i}^N \rho_{ij} \sigma_i \sigma_j \delta_i \delta_j \quad \dots\dots (5.1)$$

such that  $\delta_i \in \{0,1\}, i=1, \dots, N$

To be more specific, assume

$$\mu_i(t) = \frac{(a_i t^2 + b_i t + c_i) \lambda_{\{t \leq T_i\}} + (a_i T_i^2 + b_i T_i + c_i) \lambda_{\{t > T_i\}}}{(1+d)^t} \dots\dots (5.2)$$

Combining both the risk-adjusted (5.1) and time-adjusted (5.2) considerations, the multiple investment decision problem can be modelled as

$$\underset{\{\delta_i\}_{i=1\dots N}, t}{MAX} \quad r \sum_{i=1}^N \frac{(a_i t^2 + b_i t + c_i) \lambda_{\{t \leq T_i\}} + (a_i T_i^2 + b_i T_i + c_i) \lambda_{\{t > T_i\}}}{(1+d)^t} \delta_i - \frac{r^2}{2} \sum_{i=1}^N \sum_{j=i}^N \rho_{ij} \sigma_i \sigma_j \delta_i \delta_j \dots\dots (5.3)$$

There are two decision variables:

- $\delta_i \in \{0,1\}, i = 1, \dots, N$
- $t \geq 0$

Referring to *Appendix B* for the modified Zeus Ltd. problem that takes the time dependent expected return into account. The fund of funds manager seeks a strategy to construct the optimal portfolio in the framework of both time-adjusted and risk-adjusted objective function.

Note that this is a Mixed Integer Non Linear Programming (MINLP) model, and no existing algorithm has been identified to be able to solve the problem efficiently.

## 5.2 Problem Decomposition

The original MINLP model can be decomposed into two sub-problems:

### Sub-problem 1 (MIQP)

Fixing time  $t = t_0$



$$\underset{\{\delta_i\}_{i=1\dots N}}{MAX} \quad r \sum_{i=1}^N \mu_i(t_0) \delta_i - \frac{r^2}{2} \sum_{i=1}^N \sum_{j=i}^N \rho_{ij} \sigma_i \sigma_j \delta_i \delta_j \quad \dots\dots (5.4)$$

### Sub-problem 2 (NLP)

Fixing selected investment portfolio  $\{\delta_i\}_{i=1\dots N}$

$$\underset{t}{MAX} \quad \sum_{i=1}^N \mu_i(t) \delta_i = \sum_{i=1}^N \frac{(a_i t^2 + b_i t + c_i) \mathcal{L}_{\{t \leq T_i\}} + (a_i T_i^2 + b_i T_i + c_i) \mathcal{L}_{\{t > T_i\}}}{(1+d)^t} \delta_i \quad \dots\dots (5.5)$$

The original optimization problem has  $N + 1$  variables:  $N$  from (5.4) and 1 from (5.5).

From a graphical view, there are  $N + 2$  dimensions, and it is not possible to be visualized on paper.

However, the  $N$  dimensions from (5.4) can be transformed into 1 dimension by introducing a new variable called “*Portfolio Selection Indicator*”.

Define “*Portfolio Selection Indicator*” as  $\sum_{i=1}^N 2^{\delta_i}$ .

As a result, each distinctive portfolio selection can be represented by a distinctive “*Portfolio Selection Indicator*”.

Let

- x-axis be the “*Portfolio Selection Indicator*”:  $x = \sum_{i=1}^N 2^{\delta_i}$
- y-axis be the “*Holding Period*” (in years):  $y = t$  \dots\dots (5.6)

- z-axis be the “Expected Utility”:  $z = r \sum_{i=1}^N \mu_i(t) \delta_i - \frac{r^2}{2} \sum_{i=1}^N \sum_{j=i}^N \rho_{ij} \sigma_i \sigma_j \delta_i \delta_j$ ;

The original model can be visualized in a graphical view as in *Figure 5.1*

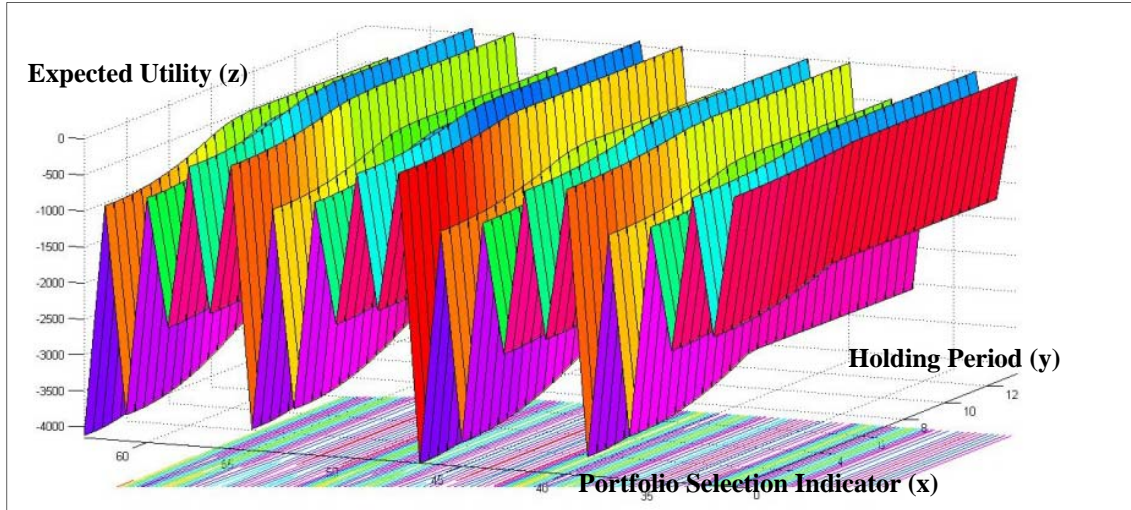


Figure 5.1 - Graphical View of Original Problem (MINLP)

The above mentioned two sub-problems are effectively projections of *Figure 5.1* onto the z-x plane and z-y plane (*Figure 5.2* & *Figure 5.3* respectively)

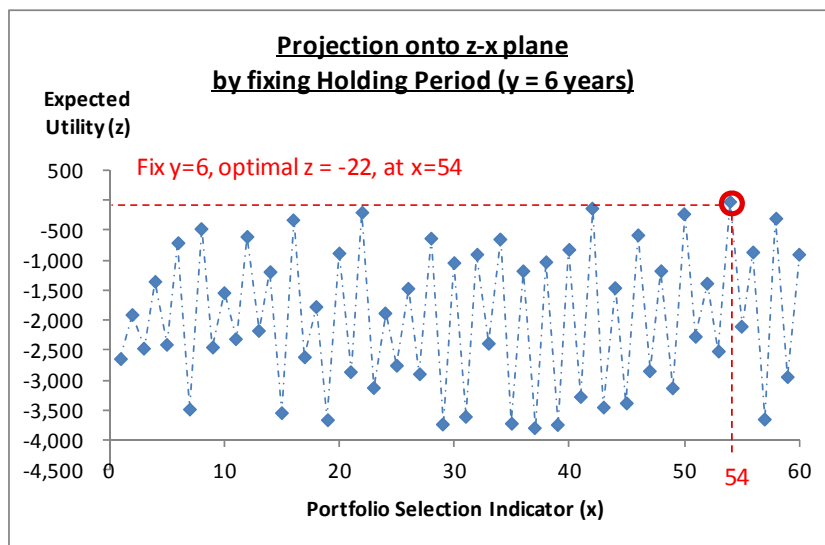


Figure 5.2 - Graphical View of Sub-Problem 1 (projection onto z-x plane)

In *Figure 5.2*, by fixing Holding Period ( $y$ ) = 6 years, the Optimal Utility ( $z$ ) = -22 at Portfolio Indicator ( $x$ ) =  $54 = \sum_{i=1}^N 2^{\delta_i}$ . It can be shown that the corresponding portfolio of 54 is  $\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6\} = \{0, 1, 1, 0, 1, 1\}$  with 2<sup>nd</sup>, 3<sup>rd</sup>, 5<sup>th</sup> and 6<sup>th</sup> investment opportunities to be chosen.

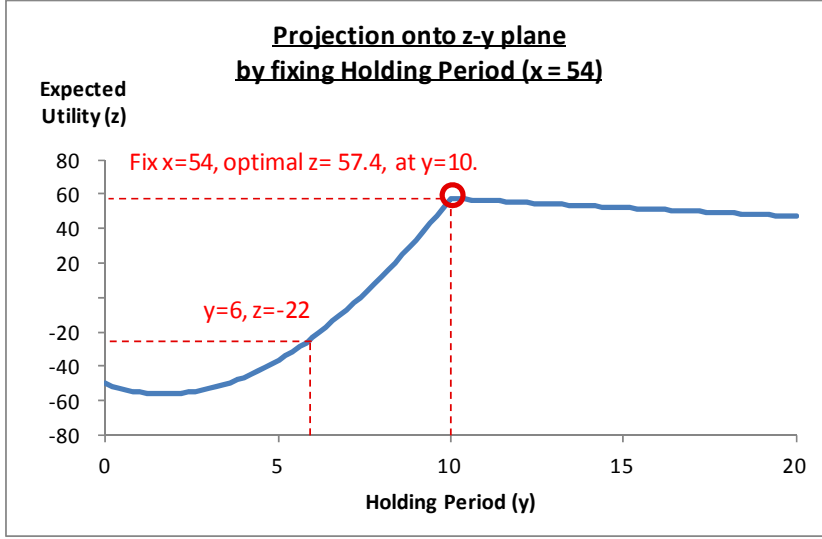


Figure 5.3 - Graphical View of Sub-Problem 2 (projection onto z-y plane)

Similarly, in *Figure 5.3*, by fixing Portfolio Indicator ( $x$ ) =  $54 = \sum_{i=1}^N 2^{\delta_i}$ , the Expected Utility ( $z$ ) can be improved by changing Holding Period ( $y$ ) from 6 years to 10 years

### 5.3 Sub-Problem 1 – Find the Best Portfolio (Fixed Time)

The first sub-problem is exactly what has been discussed in *Chapter 4*, which is an Integer Quadratic Programming problem and can be solved with the algorithms mentioned in that chapter.

It is proposed to use the exact approach (in particular CPLEX) to solve small size problems, and heuristic approach (in particular Random Restart 2-opt Hill Climbing Algorithm) to solve large size problems.

When evaluating the heuristic algorithms, 1) time efficiency; 2) probability to attain global optimums are two important considerations.

Define  $P_{(Algorithm)}$  as the probability to attain global optimal solution.

Referring to *Appendix J*, there are 100 sets of computer generated testing data for each problem size  $N = 10 \dots 60$ . Both 2-opt Hill Climbing Algorithm and CPLEX Optimization are run; and the heuristic results by the former algorithm are compared to the true global optimums.

Plot the graph of Problem Size (x-axis) vs.  $P_{2-opt (Simple)}$  (y-axis), and there is a decreasing chance to find the global optimum as the problem size grows (*Figure 5.4*).

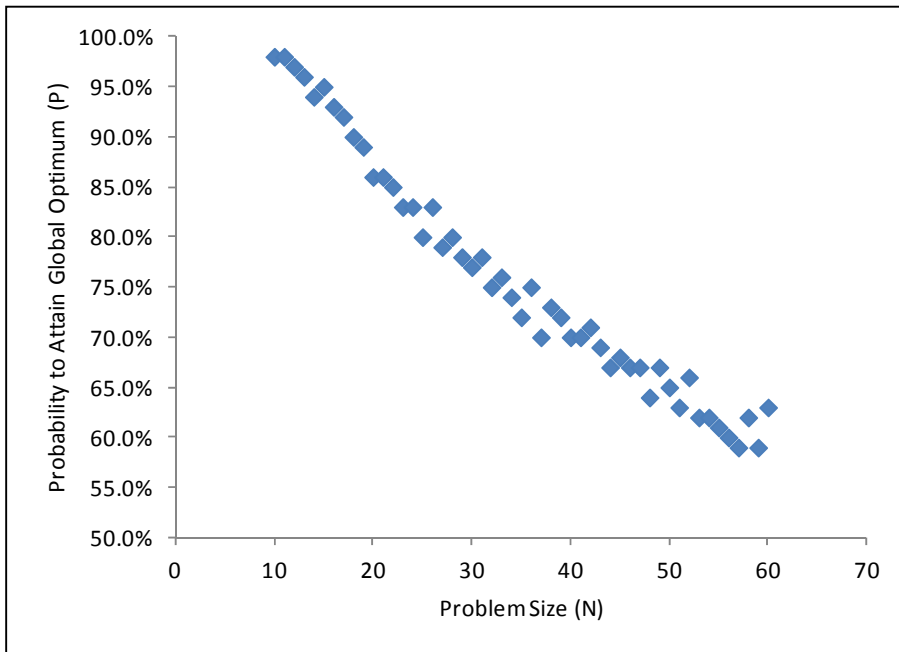


Figure 5.4 - Probability to Attain Global Optimum (2-opt Hill Climbing)

Regression Result:  $P_{2-opt (Simple)} = 70.9\% \cdot e^{-0.032 N} + 39.6\%$  with  $R^2 = 0.9694$

Assume that there is an equal chance to find the global optimum for each random starting searching point.

Recall  $M$  is the “Iteration Number” defined in *Chapter 4*.

$$P_{2-opt(M\_random)} = 1 - (1 - P_{2-opt(Simple)})^M \quad \dots\dots (5.7)$$

$$\text{or } P_{2-opt(M\_random)} = 1 - (1 - (70.9\% \cdot e^{-0.032N} + 39.6\%))^M = 1 - (60.4\% - 70.9\% \cdot e^{-0.032N})^M$$

If a minimum probability of  $P_0$  is required,

$$P_{2-opt(M\_random)} = 1 - (60.4\% - 70.9\% \cdot e^{-0.032N})^M \geq P_0 \quad \dots\dots (5.8)$$

$$\Rightarrow (60.4\% - 70.9\% \cdot e^{-0.032N})^M \leq 1 - P_0$$

$$\Rightarrow M(N, P_0) \geq \frac{\ln(1 - P_0)}{\ln(60.4\% - 70.9\% \cdot e^{-0.032N})} \quad \dots\dots (5.9)$$

$$\Rightarrow t_{\min}(N, P_0) = \min\{t_{Exhaustive-Search}(N), t_{CPLEX}(N), t_{2-opt(Random)}(N, M(N, P_0))\}$$

Recall that

$$t_{Exhaustive-Search}(N) = 0.0182 \cdot e^{0.4762N} \quad \dots\dots (5.10)$$

$$t_{CPLEX}(N) = 0.015 \cdot e^{0.3291N} \quad \dots\dots (5.11)$$

$$t_{2-opt(Random)}(N, M) \approx M \cdot t_{2-opt(Simple)}(N) = M \cdot (0.004N^2 - 0.122N + 1.280) \quad \dots\dots (5.12)$$

$$t_{2-opt(Random)}(N, P_0) = \left\lceil \frac{\ln(1 - P_0)}{\ln(60.4\% - 70.9\% \cdot e^{-0.032N})} \right\rceil \cdot (0.004N^2 - 0.122N + 1.280) \quad \dots\dots (5.13)$$

In the case of  $P_0 = 99\%$ , plot the graph for  $t_{Exhaustive-Search}$ ,  $t_{CPLEX}$ ,  $t_{2-opt(Random)}$

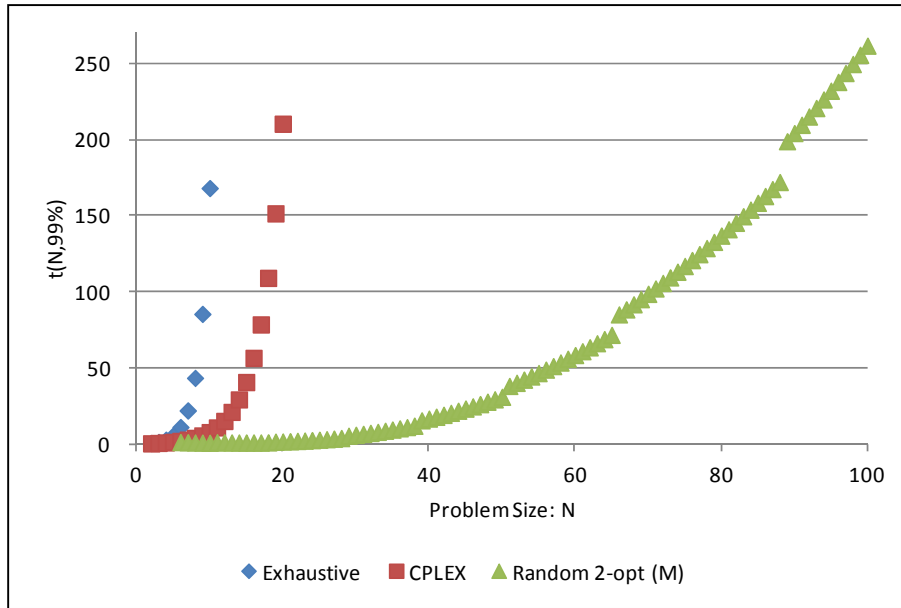


Figure 5.5 - Solving Time with Exhaustive Search, CPLEX and Random 2-opt ( $P_0 = 99\%$ )

To achieve probability to attain global optimum  $\geq 99\%$ , or  $P_{\text{(Algorithm)}} \geq 99\%$ , the preferred algorithms are summarized in Table 5.1

Table 5.1 - Different Algorithms to Use for Different Problem Sizes

Problem Size	Algorithm	$P_{\text{(Algorithm)}}$	Time (sec)
$N \leq 5$	CPLEX	100%	<1.5
$5 < N \leq 10$	Random 2-opt with $M=2$	99.2% - 99.9%	0.9 – 1.4
$10 < N \leq 18$	Random 2-opt with $M=3$	99.1% - 99.9%	1.1 – 1.3
$18 < N \leq 28$	Random 2-opt with $M=4$	99.0% - 99.8%	1.6 – 4.0
$28 < N \leq 38$	Random 2-opt with $M=5$	99.1% - 99.6%	5.5 – 12.1
$38 < N \leq 50$	Random 2-opt with $M=6$	99.0% - 99.6%	15.6 – 31.1
$50 < N \leq 65$	Random 2-opt with $M=7$	99.0% - 99.5%	38.2 – 71.8
$65 < N \leq 88$	Random 2-opt with $M=8$	99.0% - 99.4%	85.2 – 172.1
$88 < N \leq 158$	Random 2-opt with $M=9$	99.0% - 99.4%	199.9 – 736.7
$N > 158$	Random 2-opt with $M=10$	99.0% - 99.4%	>830.1

#### 5.4 Sub-Problem 2 – Find the Best Time (Fixed Portfolio)

To solve the second sub-problem, which is a Continuous Non-Linear Programming problem.

$$MAX_t \sum_{i=1}^N \mu_i(t) \delta_i = \sum_{i=1}^N \frac{(a_i t^2 + b_i t + c_i) \lambda_{\{t \leq T_i\}} + (a_i T_i^2 + b_i T_i + c_i) \lambda_{\{t > T_i\}}}{(1+d)^t} \delta_i \quad \dots\dots (5.14)$$

Its local optimums can be easily found with various built-in algorithms. Please refer to *Appendix I* for detailed program codes under Matlab environment.

#### 5.5 Heuristic Algorithm – Combination of the Two Sub-Problems

Recall that there is no efficient way to solve the Mixed Integer Non-Linear Programming model of the original problem; however, an iterative algorithm to find a heuristic solution is proposed here.

There are two sets of decision variables:  $\delta_i \in \{0,1\}, i=1,\dots,N$  and  $t \in (0,+\infty)$

Define

$$f(\{\delta_i\}, t) = r \sum_{i=1}^N \frac{(a_i t^2 + b_i t + c_i) \lambda_{\{t \leq T_i\}} + (a_i T_i^2 + b_i T_i + c_i) \lambda_{\{t > T_i\}}}{(1+d)^t} \delta_i - \frac{r^2}{2} \sum_{i=1}^N \sum_{j=i}^N \rho_{ij} \sigma_i \sigma_j \delta_i \delta_j$$

1) to fix the first variable and solve for the optimal solution with the second one;

$$\bar{t} = t^*(\{\delta_i\}) = \arg \{ \max_t (f(\{\delta_i\}, t)) \} \quad \dots\dots (5.15)$$

2) to fix the second variable and solve for the first one.

$$\{\bar{\delta}_i\} = \{\delta_i\}^*(\bar{t}) = \arg \{ \max_{\{\delta_i\}} (f(\{\delta_i\}, \bar{t})) \} \quad \dots\dots (5.16)$$

3) Iteratively, this algorithm will continuously improve the expected utility function,

until

$$\{\delta_i\}^*(t^*) = \arg\{\max_{\{\delta_i\}} (f(\{\delta_i\}, t^*(\{\delta_i\}^*)))\} \text{ and } t^*(\{\delta_i\}^*) = \arg\{\max_t (f(\{\delta_i\}^*(t^*), t))\}$$

..... (5.17)

Algorithm 5.1 (Iterative Algorithm to solve MINLP):

initialize  $\{\delta_i\}$

do {

$$\overline{\{\delta_i\}} = \{\delta_i\};$$

$$\bar{t} = t^*(\overline{\{\delta_i\}}) = \arg\{\max_t (f(\overline{\{\delta_i\}}, t))\}; \quad // \text{second sub-problem}$$

$$\{\delta_i\} = \{\delta_i\}^*(\bar{t}) = \arg\{\max_{\{\delta_i\}} (f(\{\delta_i\}, \bar{t}))\}; \quad // \text{first sub-problem}$$

} while ( $\{\delta_i\} \neq \overline{\{\delta_i\}}$ )

END

Graphically, refer to *Figure 5.2* and *Figure 5.3*. Each iteration performs maximization operation along x-axis ( $\{\delta_i\}_{i=1 \dots N}$ ) and y-axis ( $t$ ) alternatively.

There are two things to note: firstly, this algorithm will not run into a loop, because the objective function will keep improving and it must stop at some point. Furthermore, it converges to a local optimum solution: a point which is the global optimum along both x-axis and y-axis.

However, the algorithm does not guarantee a global optimal point, and



Figure 5.6 shows an example where local optimal point A can no longer be improved neither along x-axis or along y-axis, although point B is clearly dominating it.

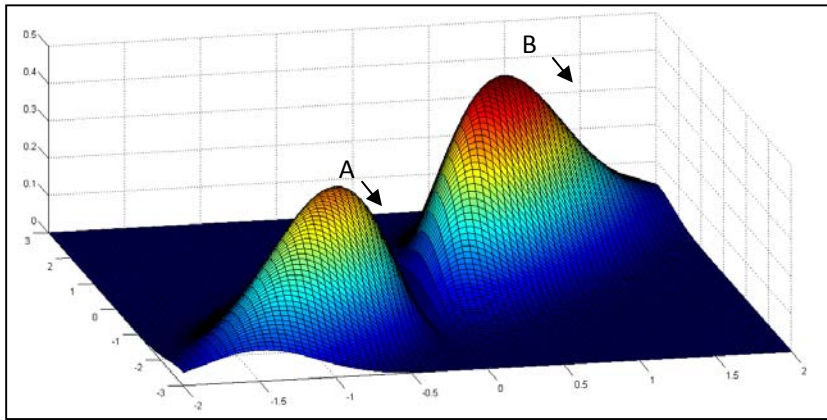


Figure 5.6 - Iterative Algorithm: Local Optimum Example

Finally, apply this Iterative Algorithm to the modified Zeus Ltd. Problem (*Appendix B*) mentioned at the beginning of this Chapter.

As a result of the previous studies, different heuristic algorithms should be used to solve the first sub-problem depending on the problem sizes. Below is a summarizing graph of Solving Time ( $t$ ) plotted against Problem Size ( $N$ ).

Recall  $M$  is the “Iteration Number” defined in Chapter 4.

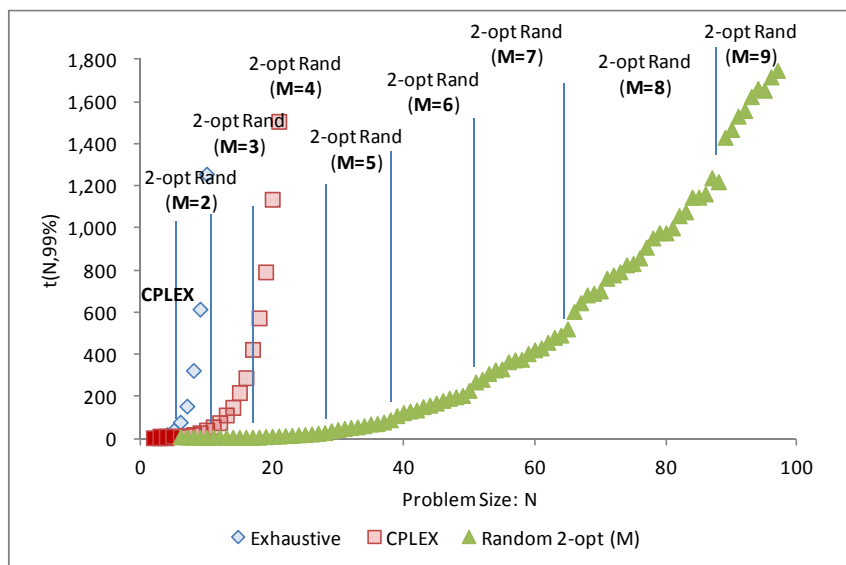


Figure 5.7 - Solving Time depending on different Problem Sizes ( $P_0 = 99\%$ )

## Chapter 6 Conclusion

### 6.1 Contribution

In this thesis, the risk-adjusted and time-adjusted discrete asset allocation problem facing many private equity fund of funds managers is studied. The difficulty arises from the considerations of both “optimal portfolio construction” and “best holding period” at the same time, which resulted in a Mixed Integer Non Linear Programming (MINLP) model. To the best of the author’s knowledge, there is no efficient and effective way to solve this problem. Several heuristic algorithms are proposed here to balance between time efficiency and accuracy with respect to the optimal solution.

Firstly, one starts with single investment decision without time consideration. It was shown that with the presence of correlation among investment candidates and initial wealth, the “Delta Property” no longer holds for CARA utility function. The “relaxed Delta Property” is then proposed, where the investment decision is independent of the expected return of the initial portfolio.

Secondly, multiple investment opportunities decision problem is then studied, still without time consideration. It can be shown that the exact algorithms (Exhaustive Search Algorithm and CPLEX) are not efficient as problem size becomes big; and several heuristic algorithms are proposed including Greedy Algorithm, k-opt Hill Climbing Algorithm, Random Restarted k-opt Hill Climbing Algorithm, and Stochastic Gradient Ascent Algorithm. The comparison among various heuristic algorithms suggests that the “Random Restarted k-opt Hill Climbing Algorithm” (with  $k=2$  or  $3$ ) seems to be a good compromise between solving time and closeness to the true optimal solution.

Lastly, the time value is introduced into the model and it becomes a Mixed Integer Non Linear Programming (MINLP) problem. While directly solving the model seems difficult, it can be decomposed into two sub-problems; and the “Iterative Algorithm” solves both sub-models iteratively and eventually provided a heuristic solution. In addition, depending on different problem sizes, a check-list table is presented to choose the best heuristic algorithm in order to solve the model the most efficiently, yet at the same time to guarantee some amount of accuracy to the true optimal solution.

Recall that the “private equity fund investment decision problem” described at the very beginning of this thesis has the following characteristics: certain/uncertain initial wealth, discrete investment amount, and target to maximize expected utility.

This thesis transforms the problem into a Mixed Integer Non Linear Programming model, and heuristically solved it using a modified Hill Climbing Algorithm.

As mentioned in the beginning, three topics were covered in this thesis: *Decision Analysis*, *Portfolio Diversification* and *Optimization Algorithm*. There have been various studies done on each of the topics, and *Table 6.1* summarizes the problem characteristics covered by different research works, as well as the tools used by them.

As noticed, none of the studies can be directly applied to problem with all the above mentioned characteristics; while this thesis bridges the gaps among the studies and successfully solves the problem.

Table 6.1 - Summary of Other Major Research Works

Research Papers	Decision Analysis		Portfolio Diversification		Optimization Algorithm	
	Certain Initial Wealth	Uncertain Initial Portfolio	Maxi'n of Expected Utility	Discrete Inv't Amount	MINLP	Hill Climbing Algorithm
<b>This Thesis</b>	✓	✓	✓	✓	✓	✓
Neumann & Morgenstern, 1944	✓		✓			
Ronald, 1964	✓		✓			
Arrow & Pratt, 1965	✓					
Pfanzagl, 1959	✓					
Howard & Raiffa, 1968	✓					
Cohen & Pogue, 1967	✓		✓			
Perold, 1984	✓		✓			
Ghasemzadeh, Archer & Iyogun, 1999	✓		✓	✓		
Longstaff, 2001	✓		✓	✓		
Browne, Milevsky & Salisbury, 2003	✓		✓	✓		
Patel & Subrahmanya, 1982	✓		✓			
Perez & Malley, 1983	✓		✓			
Amit & Livnat, 1989	✓		✓			
Kritzman, 1992	✓		✓			
Gomes & Michaelides, 2005	✓		✓			
Peng, 1980						✓
Goldfeld, Quandt & Trotter, 1966						✓
Russell & Norvig, 2003						✓
Fouskakis & Draper, 2002						✓

## **6.2 Limitation**

The main limitation of this thesis is that the “Iterative Algorithm” only provides a heuristic approach to solve the MINLP problem, and no conclusion has been drawn on the closeness of its heuristic solution to the true global optimal value. In addition, the check list of when to use which algorithm is derived from the optimization of the first sub-problem (IQP), and it may not be the best choice for the combined MINLP model.

## **6.3 Future Work**

For future work, it is proposed to extend the check-list table beyond the true optimal probability of 99%. Other levels of optimal probability should also be studied to complete the list of when to use which algorithm. Furthermore, although heuristic solutions are achievable with the “Iterative Algorithm”, the “Random Restart” version can also be implemented to move from one local optimal solution to other local optimal solutions, and a further improvement could be achieved with certain probabilities. Lastly, the application of proposed algorithms has not yet been generalized for other portfolio allocation problems. Additional work could be done to broaden the application scope.

## Bibliography

- Amit, R. & Livnat, J. (1989). *Efficient Corporate Diversification: Methods and Implications*. Management Science, Vol. 35, No. 7 (Jul., 1989), pp. 879-897
- Ankrim, E. M. & Hensel, C. R. (1993). *Commodities in Asset Allocation: A Real-Asset Alternative to Real Estate?* Financial Analysts Journal, Vol. 49, No. 3 (May - Jun., 1993), pp. 20-29
- Arnott, R. D. & von Germeten, J. N. (1983). *Systematic Asset Allocation*. Financial Analysts Journal, Vol. 39, No. 6 (Nov. - Dec., 1983), pp. 31-38
- Bazarra, M. S. & Shetty, C. M. (1979). *Nonlinear Programming. Theory and Algorithms*. New York: John Wiley and Sons.
- Best, M. J. & Hlouskova, J. (2005). *An Algorithm for Portfolio Optimization with Transaction Costs*. Management Science, Vol. 51, No. 11 (Nov., 2005), pp. 1676-1688
- Brandão, L. E. & Dyer, J. S. (2005). *Decision Analysis and Real Options: A Discrete Time Approach to Real Option Valuation*. Annals of Operation Research, Vol. 135, pp. 21-39
- Brandão, L. E., Dyer, J. S. & Hahn, W. J. (2005). *Using Binomial Decision Trees to Solve Real-Option Valuation Problems*. Decision Analysis, Vol. 2, No. 2, (Jun 2005), pp. 69-88
- Browne, S., Milevsky, M. A. & Salisbury, T. S. (2003). *Asset Allocation and the Liquidity Premium for Illiquid*. The Journal of Risk and Insurance, Vol. 70, No. 3 (Sep., 2003), pp. 509-526
- Cavaglia, S. & Moroz, V. (2002). *Cross-Industry, Cross-Country Allocation*. Financial

- Analysts Journal, Vol. 58, No. 6 (Nov. - Dec., 2002), pp. 78-97
- Chen, P., Ibbotson, R. G., Milevsky, M. A. & Zhu, K. X. (2006). *Human Capital, Asset Allocation, and Life Insurance*. Financial Analysts Journal, Vol. 62, No. 1 (Jan. - Feb., 2006), pp. 97-109
- Cohen, K. J. & Pogue, J. A. (1967). *An Empirical Evaluation of Alternative Portfolio-Selection Models*. The Journal of Business, Vol. 40, No. 2 (Apr., 1967), pp. 166-193
- Cvitanić, J., Lazrak, A., Martellini, L. & Zapatero, F. (2006). *Dynamic Portfolio Choice with Parameter Uncertainty and the Economic Value of Analysts' Recommendations*. The Review of Financial Studies, Vol. 19, No. 4 (Winter, 2006), pp. 1113-1156
- Dantzig, G. B. (1963). *Linear Programming and Extensions*. Princeton University Press: New Jersey
- Elliott, R. & Hinz, J. (2004). *Portfolio optimization, hidden Markov models, and technical analysis of P&F-Charts*. Tübingen: Universität Tübingen
- Eun, C. S. & Resnick, B. G. (1994). *International Diversification of Investment Portfolios: U.S. and Japanese Perspectives*. Management Science, Vol. 40, No. 1, Focused Issue: Is Management Science
- Fouskakis, D. & Draper, D. (2002). *Stochastic Optimization: A Review*. International Statistical Review / Revue Internationale de Statistique, Vol. 70, No. 3(Dec., 2002), pp. 315-349
- Ghasemzadeh, F., Archer, N. & Iyogun, P. (1999). *A Zero-One Model for Project Portfolio Selection and Scheduling*. The Journal of the Operational Research Society, Vol. 50, No. 7 (Jul., 1999), pp. 745-755

- Gomes, F. & Michaelides, A. (2005). *Optimal Life-Cycle Asset Allocation: Understanding the Empirical Evidence*. The Journal of Finance, Vol. 60, No. 2 (Apr., 2005), pp. 869-904
- Goldfarb, D. & Iyengar, G. (2003). *Robust Portfolio Selection Problems*. Mathematics of Operations Research, Vol. 28, No. 1 (Feb., 2003), pp. 1-38
- Goldfeld, S. M., Quandt, R. E. & Trotter, H. F. (1966). *Maximization by Quadratic Hill-Climbing*. Econometrica, Vol. 34, No. 3 (Jul., 1966), pp. 541-551
- Guindon, S. & Gascuel, O. (2003). *A Simple, Fast, and Accurate Algorithm to Estimate Large Phylogenies by Maximum Likelihood*. Systematic Biology, Vol. 52, No. 5 (Oct., 2003), pp. 696-704
- Hajek, B. (1986). *Optimization by Simulated Annealing: A Necessary and Sufficient Condition for Convergence*. Lecture Notes-Monograph Series, Vol. 8, Adaptive Statistical Procedures and Related Topics (1986), pp. 417-427
- Haring, J. E. & Smith, G. C. (1959). *Utility Theory, Decision Theory, and Profit Maximization*. The American Economic Review, Vol. 49, No. 4 (Sep., 1959), pp. 566-583
- Holland, J. H. (1975). *Adaptation in Natural and Artificial Systems*. Michigan: University of Michigan Press
- Howard, R. A. (1966). *Decision analysis: Applied decision theory*. Proceedings of the 4th International Conference on Operational Research, pp. 55-77.
- John, S. (2001). *Risk and decision analysis in projects* (2<sup>nd</sup> ed.). Pennsylvania: Project management institute
- Johnson, D. S. & McGeoch, L. A. (1995). *The Traveling Salesman Problem: A Case Study in Local Optimization*. Amherst: Amherst College



- Kantorovich, L. V. (1940). *A New Method of Solving Some Classes of External Problems*. Doklady Akad Sci USSR, Vol. 28 (1940), pp. 211-214
- Kim, H. Y. & Viens, F. G. (2010). *Portfolio Optimization in Discrete Time with Proportional Transaction Costs under Stochastic Volatility*. Indiana: Purdue University.
- Kritzman, M. (1992), *Asset Allocation for Individual Investors*. Financial Analysts Journal, Vol. 48, No. 1 (Jan. - Feb., 1992), pp. 12-13
- Longstaff, F. A. (2001). *Optimal Portfolio Choice and the Valuation of Illiquid Securities*. The Review of Financial Studies, Vol. 14, No. 2 (Summer, 2001), pp. 407-431
- Markowitz, H. M. (1952). *Portfolio selection*. The Journal of Finance, Vol. 7, No. 1 (Mar., 1952), pp. 77-91
- Markowitz, H. M. (1959). *Portfolio selection: efficient diversification of investments*. London: Chapman & Hall, Ltd.
- Matyas, J. (1965). *Random Optimization*. Automation and Remote Control. Vol. 26, No. 2, pp. 246-253
- Murty, K. G. (1988). *Linear Complementarity, Linear and Nonlinear Programming*. Berlin: Heldermann Verlag.
- Nemhauser, G. L. & Wolsey, L. A. (1988). *Integer and Combinatorial Optimization*. New York: John Wiley and Sons.
- Pang, J. S. (1980). *A New and Efficient Algorithm for a Class of Portfolio Selection Problems*. Operations Research, Vol. 28, No. 3, Part 2 (May - Jun., 1980), pp. 754-767

- Patel, N. R. & Subrahmanya, M. G. (1982). *A Simple Algorithm for Optimal Portfolio Selection with Fixed Transaction Costs*. Management Science, Vol. 28, No. 3 (Mar., 1982), pp. 303-314
- Perez, R. & Malley, S. (1983). *Asset Allocation and the Social Security System*. Financial Management, Vol. 12, No. 1 (Spring, 1983), pp. 29-35
- Perold, A. F. (1984). *Large-Scale Portfolio Optimization*. Management Science, Vol. 30, No. 10 (Oct., 1984), pp. 1143-1160
- Puelz, A. V. (2002). *A Stochastic Convergence Model for Portfolio Selection*. Operations Research, Vol. 50, No. 3 (May - Jun., 2002), pp. 462-476
- Robin, H. & Monro, S. (1951). *A Stochastic Approximation Method*. Annals of Mathematical Statistics. Vol. 22, No. 3, pp. 400-407
- Russell, S. J. & Norvig, P. (2003). *Artificial Intelligence: A Modern Approach*, (3<sup>rd</sup> ed.). New Jersey: Prentice Hall.
- Sefton, J. A. (2010). *Discrete-Time Dynamic Portfolio Optimisation with Trading costs*. Imperial College Business School: London
- Sharpe, W. F. (1987). *Integrated Asset Allocation*. Financial Analysts Journal, Vol. 43, No. 5 (Sep. - Oct., 1987), pp. 25-32
- Smith, J. E. & Nau, R. F. (1995). *Valuing Risky Projects: Option Pricing Theory and Decision Analysis*. Management Science, Vol. 41, No. 5 (May, 1995), pp. 795-816
- Smith, J. E. & McCardle, K. F. (1998). *Valuing Oil Properties: Integrating Option Pricing and Decision Analysis Approaches*. Operations Research, Vol. 46, No. 2 (Mar – Apr., 1998), pp. 198-218

- Smith, J. E. (2005). *Alternative Approaches for Solving Real-Options Problems*. Decision Analysis, Vol. 2, No. 2, (Jun 2005), pp. 89–102
- Storn, R. & Price, K. (1997). *Differential Evolution - a Simple and Efficient Heuristic for Global Optimization over Continuous Spaces*. Journal of Global Optimization Vol. 11, No. 4, pp. 341-359
- Tilley, J. A. & Latainer, G. D. (1985). *A Synthetic Option Framework for Asset Allocation*. Financial Analysts Journal, Vol. 41, No. 3 (Jun. - Jul., 1985), pp. 32-43

## Appendix A – Case Study (Risk-Adjusted): Zeus Ltd.

Unit: USD million

	Expected Return ( $\mu_i$ with $i=1...60$ )	Standard Deviation ( $\sigma_i$ with $i=1...60$ )
1	49.81	4.41
2	50.09	5.45
3	66.58	10.05
4	49.96	12.65
5	50.00	5.13
6	49.85	4.46
7	87.07	14.51
8	120.22	21.94
9	56.15	7.67
10	42.68	2.61
11	49.72	4.32
12	126.62	23.42
13	47.02	3.35
14	50.14	5.54
15	49.88	4.49
16	49.77	4.37
17	29.66	1.35
18	45.38	3.03
19	50.17	5.57
20	50.00	5.03
21	53.84	15.68
22	50.09	5.45
23	53.83	7.07
24	72.89	11.43
25	44.53	2.88
26	56.44	16.77
27	49.76	4.36
28	51.52	6.34
29	195.39	71.35
30	31.30	6.48
31	69.25	10.64
32	88.43	14.81
33	50.20	5.60
34	55.15	7.42
35	50.14	5.54
36	49.87	12.46
37	50.00	4.90
38	34.68	7.30
39	52.98	15.30
40	47.33	3.42
41	96.77	16.65
42	55.36	7.47
43	49.69	4.30
44	50.00	5.05
45	76.40	12.19
46	57.68	8.04
47	81.72	13.35
48	98.40	17.01
49	57.94	8.11
50	34.65	1.74
51	52.65	6.72
52	49.59	4.22
53	50.69	5.96
54	66.74	10.09
55	49.79	4.39
56	49.33	11.96
57	50.00	4.97
58	50.01	5.17
59	105.80	18.67
60	121.20	22.17

Correlation (  $\rho_{ij}$  with  $i=1\ldots 20, j=1\ldots 60$ )

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1.00	-0.13	0.23	-0.15	0.73	0.50	0.18	-0.23	0.53	0.91	-0.11	0.06	0.78	0.68	-0.05	0.35	0.59	0.81	0.23	0.68
2	-0.13	1.00	0.27	0.68	-0.48	-0.13	0.14	0.17	-0.62	-0.31	-0.07	0.41	0.19	-0.42	-0.46	0.33	-0.29	-0.29	0.17	-0.25
3	0.23	0.27	1.00	0.56	0.31	0.32	0.39	0.52	0.06	0.15	-0.16	0.73	0.46	0.33	0.34	0.75	-0.24	0.37	0.55	0.41
4	-0.15	0.68	0.56	1.00	-0.23	0.02	0.51	0.66	-0.62	-0.38	-0.17	0.88	0.28	-0.30	-0.12	0.76	-0.65	-0.08	0.46	-0.08
5	0.73	-0.48	0.31	-0.23	1.00	0.68	0.37	0.14	0.76	0.67	0.26	0.09	0.42	0.85	0.46	0.33	0.43	0.88	0.48	0.77
6	0.50	-0.13	0.32	0.02	0.68	1.00	0.39	0.17	0.31	0.34	0.40	0.22	0.32	0.72	0.13	0.39	0.27	0.59	0.75	0.68
7	0.18	0.14	0.39	0.51	0.37	0.39	1.00	0.76	-0.06	-0.06	0.36	0.58	0.14	0.16	0.30	0.62	-0.13	0.53	0.61	0.22
8	-0.23	0.17	0.52	0.66	0.14	0.17	0.76	1.00	-0.16	-0.42	0.31	0.71	-0.11	0.00	0.51	0.61	-0.59	0.15	0.56	0.08
9	0.53	-0.62	0.06	-0.62	0.76	0.31	-0.06	-0.16	1.00	0.65	0.29	-0.33	0.12	0.75	0.57	-0.16	0.54	0.63	0.04	0.60
10	0.91	-0.31	0.15	-0.38	0.67	0.34	-0.06	-0.42	0.65	1.00	-0.22	-0.14	0.71	0.67	0.03	0.16	0.70	0.73	-0.01	0.58
11	-0.11	-0.07	-0.16	-0.17	0.26	0.40	0.36	0.31	0.29	-0.22	1.00	-0.19	-0.46	0.22	0.33	-0.19	0.14	0.19	0.36	0.14
12	0.06	0.41	0.73	0.88	0.09	0.22	0.58	0.71	-0.33	-0.14	-0.19	1.00	0.43	0.08	0.11	0.89	-0.59	0.16	0.62	0.18
13	0.78	0.19	0.46	0.28	0.42	0.32	0.14	-0.11	0.12	0.71	-0.46	0.43	1.00	0.40	-0.24	0.62	0.26	0.54	0.22	0.45
14	0.68	-0.42	0.33	-0.30	0.85	0.72	0.16	0.00	0.75	0.67	0.22	0.08	0.40	1.00	0.39	0.26	0.43	0.71	0.50	0.81
15	-0.05	-0.46	0.34	-0.12	0.46	0.13	0.30	0.51	0.57	0.03	0.33	0.11	-0.24	0.39	1.00	0.13	-0.13	0.35	0.21	0.28
16	0.35	0.33	0.75	0.76	0.33	0.39	0.62	0.61	-0.16	0.16	-0.19	0.89	0.62	0.26	0.13	1.00	-0.29	0.43	0.65	0.32
17	0.59	-0.29	-0.24	-0.65	0.43	0.27	-0.13	-0.59	0.54	0.70	0.14	-0.59	0.26	0.43	-0.13	-0.29	1.00	0.51	-0.14	0.29
18	0.81	-0.29	0.37	-0.08	0.88	0.59	0.53	0.15	0.63	0.73	0.19	0.16	0.54	0.71	0.35	0.43	0.51	1.00	0.44	0.67
19	0.23	0.17	0.55	0.46	0.48	0.75	0.61	0.56	0.04	-0.01	0.36	0.62	0.22	0.50	0.21	0.65	-0.14	0.44	1.00	0.44
20	0.68	-0.25	0.41	-0.08	0.77	0.68	0.22	0.08	0.60	0.58	0.14	0.18	0.45	0.81	0.28	0.32	0.29	0.67	0.44	1.00
21	-0.26	0.47	0.50	0.79	-0.04	0.14	0.71	0.86	-0.41	-0.51	0.22	0.68	-0.07	-0.22	0.25	0.61	-0.56	0.04	0.58	-0.05
22	-0.48	-0.14	0.18	0.13	0.00	0.00	0.40	0.68	0.08	-0.48	0.51	0.20	-0.55	0.01	0.67	0.04	-0.37	-0.08	0.21	-0.02
23	0.78	-0.48	0.07	-0.54	0.84	0.61	-0.01	-0.30	0.83	0.80	0.22	-0.27	0.38	0.86	0.24	0.01	0.71	0.74	0.22	0.72
24	-0.40	0.16	0.24	0.42	0.04	0.22	0.65	0.84	-0.14	-0.58	0.63	0.38	-0.44	-0.04	0.47	0.24	-0.43	0.03	0.52	0.02
25	0.51	-0.15	0.39	0.04	0.74	0.63	0.49	0.35	0.56	0.39	0.43	0.31	0.30	0.72	0.43	0.43	0.19	0.72	0.63	0.58
26	0.04	0.44	0.56	0.86	0.12	0.30	0.76	0.73	-0.43	-0.22	0.00	0.87	0.33	-0.05	0.02	0.84	-0.46	0.24	0.70	0.05
27	0.41	-0.45	-0.23	-0.74	0.55	0.21	-0.22	-0.37	0.84	0.51	0.41	-0.58	-0.05	0.56	0.32	-0.42	0.64	0.44	-0.08	0.38
28	0.27	-0.30	0.31	-0.01	0.69	0.70	0.58	0.54	0.44	0.11	0.55	0.25	-0.02	0.67	0.56	0.31	0.06	0.55	0.70	0.63
29	0.11	0.69	0.69	0.89	-0.08	0.14	0.38	0.47	-0.47	-0.11	-0.29	0.81	0.52	-0.11	-0.19	0.79	-0.41	0.05	0.45	0.15
30	0.36	0.20	0.11	0.00	0.22	0.40	0.14	-0.10	0.04	0.28	0.13	0.02	0.25	0.34	-0.18	0.21	0.33	0.29	0.31	0.27
31	0.64	-0.63	-0.02	-0.68	0.71	0.27	-0.21	-0.41	0.88	0.80	0.04	-0.38	0.29	0.70	0.33	-0.16	0.62	0.62	-0.09	0.50
32	0.89	-0.16	0.21	-0.22	0.78	0.63	0.27	-0.21	0.56	0.80	0.11	-0.02	0.62	0.72	0.00	0.27	0.71	0.85	0.40	0.63
33	0.51	-0.47	0.32	-0.25	0.65	0.12	0.23	0.11	0.72	0.64	-0.01	0.08	0.33	0.55	0.58	0.21	0.27	0.70	0.02	0.40
34	0.33	-0.19	0.32	0.13	0.64	0.71	0.77	0.55	0.34	0.16	0.56	0.35	0.10	0.56	0.40	0.42	0.11	0.66	0.65	0.54
35	0.89	-0.16	0.38	0.01	0.84	0.68	0.41	0.08	0.56	0.75	0.10	0.27	0.70	0.77	0.14	0.50	0.44	0.89	0.48	0.77
36	0.24	0.16	0.12	-0.05	-0.11	-0.15	-0.19	-0.30	0.03	0.30	-0.32	-0.08	0.29	0.01	-0.32	-0.07	0.25	0.04	-0.34	0.24
37	0.41	-0.42	0.21	-0.54	0.55	0.16	-0.20	-0.23	0.80	0.61	0.07	-0.23	0.19	0.63	0.48	-0.09	0.45	0.45	-0.06	0.40
38	0.02	0.59	0.50	0.94	-0.13	0.07	0.54	0.58	-0.60	-0.22	-0.27	0.88	0.42	-0.22	-0.20	0.81	-0.58	0.01	0.45	-0.05
39	0.17	0.67	0.64	0.93	-0.05	0.14	0.51	0.53	-0.48	-0.09	-0.27	0.85	0.53	-0.13	-0.19	0.82	-0.43	0.13	0.46	0.15
40	0.59	0.06	0.57	0.44	0.67	0.58	0.81	0.55	0.22	0.36	0.17	0.63	0.53	0.50	0.28	0.75	0.02	0.79	0.70	0.56
41	0.35	0.35	0.63	0.72	0.29	0.43	0.55	0.50	-0.19	0.12	-0.19	0.89	0.60	0.30	-0.06	0.86	-0.35	0.34	0.67	0.39
42	0.23	0.16	0.74	0.69	0.41	0.47	0.74	0.79	-0.05	-0.01	0.02	0.88	0.38	0.35	0.34	0.86	-0.39	0.44	0.74	0.46
43	0.79	-0.16	0.45	-0.17	0.70	0.49	0.05	-0.18	0.62	0.83	-0.13	0.13	0.70	0.79	0.16	0.38	0.51	0.69	0.28	0.63
44	0.26	-0.46	0.41	-0.29	0.61	0.41	0.26	0.26	0.65	0.36	0.28	0.03	0.04	0.69	0.68	0.15	0.27	0.50	0.25	0.52
45	-0.06	0.21	0.67	0.67	0.27	0.41	0.72	0.85	-0.13	-0.27	0.20	0.77	0.09	0.20	0.43	0.73	-0.46	0.26	0.77	0.26
46	0.35	0.01	0.77	0.41	0.53	0.45	0.45	0.58	0.34	0.23	0.06	0.67	0.40	0.58	0.48	0.67	-0.18	0.50	0.60	0.70
47	-0.01	0.24	0.61	0.75	0.24	0.33	0.83	0.84	-0.23	-0.22	0.15	0.85	0.17	0.09	0.31	0.80	-0.48	0.31	0.72	0.09
48	0.06	0.38	0.65	0.76	0.21	0.49	0.63	0.65	-0.31	-0.19	0.05	0.83	0.29	0.15	0.09	0.80	-0.43	0.23	0.83	0.19
49	0.57	-0.45	0.09	-0.47	0.78	0.59	0.15	-0.02	0.74	0.55	0.35	-0.16	0.18	0.83	0.36	0.02	0.52	0.63	0.31	0.71
50	0.83	-0.35	0.19	-0.38	0.73	0.42	0.11	-0.22	0.71	0.86	0.00	-0.17	0.52	0.70	0.22	0.12	0.69	0.80	0.06	0.72
51	-0.40	-0.16	0.09	0.11	0.09	0.14	0.51	0.71	0.04	-0.49	0.58	0.17	-0.55	0.07	0.58	0.04	-0.32	-0.02	0.30	0.11
52	0.68	-0.19	0.53	0.14	0.81	0.76	0.43	0.30	0.49	0.53	0.14	0.47	0.55	0.85	0.29	0.59	0.16	0.74	0.68	0.86
53	0.16	-0.09	0.35	0.17	0.56	0.49	0.65	0.60	0.31	0.02	0.52	0.33	-0.03	0.43	0.58	0.45	-0.03	0.51	0.68	0.23
54	0.30	0.20	0.41	0.50	0.44	0.47	0.92	0.65	-0.01	0.04	0.39	0.55	0.23	0.21	0.22	0.64	-0.05	0.63	0.71	0.26
55	0.77	-0.12	0.57	0.19	0.85	0.71	0.52	0.29	0.47	0.60	0.07	0.50	0.65	0.79	0.28	0.68	0.21	0.83	0.70	0.77
56	0.19	0.49	0.81	0.81	0.11	0.24	0.45	0.50	-0.27	0.01	-0.27	0.85	0.54	0.07	0.00	0.82	-0.36	0.27	0.54	0.27
57	0.47	-0.52	0.11	-0.44	0.77	0.62	0.23	0.08	0.78	0.44	0.54	-0.21	0.03	0.74	0.49	-0.05	0.49	0.68	0.33	0.71
58	0.71	-0.37	0.32	-0.27	0.85	0.65	0.40	0.11	0.74	0.66	0.32	0.02	0.34	0.86	0.44	0.25	0.54	0.85	0.42	0.80
59	0.04	-0.07	0.52	0.39	0.43	0.45	0.72	0.84	0.16	-0.15	0.40	0.55	-0.02	0.36	0.58	0.51	-0.29	0.40	0.61	0.48
60	0.43	0.31	0.79	0.63	0.39	0.47	0.50	0.45	-0.02	0.23	-0.20	0.84	0.62	0.44	0.09	0.85	-0.21	0.45	0.69	0.54

# Correlation ( $\rho_{ij}$ with $i=21\dots40, j=1\dots60$ )

	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	-0.26	-0.48	0.78	-0.40	0.51	0.04	0.41	0.27	0.11	0.36	0.64	0.89	0.51	0.33	0.89	0.24	0.41	0.02	0.17	0.59
2	0.47	-0.14	-0.48	0.16	-0.15	0.44	-0.45	-0.30	0.69	0.20	-0.63	-0.16	-0.47	-0.19	-0.16	0.16	-0.42	0.59	0.67	0.06
3	0.50	0.18	0.07	0.24	0.39	0.56	-0.23	0.31	0.69	0.11	-0.02	0.21	0.32	0.32	0.38	0.12	0.21	0.50	0.64	0.57
4	0.79	0.13	-0.54	0.42	0.04	0.86	-0.74	-0.01	0.89	0.00	-0.68	-0.22	-0.25	0.13	0.01	-0.05	-0.54	0.94	0.93	0.44
5	-0.04	0.00	0.84	0.04	0.74	0.12	0.55	0.69	-0.08	0.22	0.71	0.78	0.65	0.64	0.84	-0.11	0.55	-0.13	-0.05	0.67
6	0.14	0.00	0.61	0.22	0.63	0.30	0.21	0.70	0.14	0.40	0.27	0.63	0.12	0.71	0.68	-0.15	0.16	0.07	0.14	0.58
7	0.71	0.40	-0.01	0.65	0.49	0.76	-0.22	0.58	0.38	0.14	-0.21	0.27	0.23	0.77	0.41	-0.19	-0.20	0.54	0.51	0.81
8	0.86	0.68	-0.30	0.84	0.35	0.73	-0.37	0.54	0.47	-0.10	-0.41	-0.21	0.11	0.55	0.08	-0.30	-0.23	0.58	0.53	0.55
9	-0.41	0.08	0.83	-0.14	0.56	-0.43	0.84	0.44	-0.47	0.04	0.88	0.56	0.72	0.34	0.56	0.03	0.80	-0.60	-0.48	0.22
10	-0.51	-0.48	0.80	-0.58	0.39	-0.22	0.51	0.11	-0.11	0.28	0.80	0.80	0.64	0.16	0.75	0.30	0.61	-0.22	-0.09	0.36
11	0.22	0.51	0.22	0.63	0.43	0.00	0.41	0.55	-0.29	0.13	0.04	0.11	-0.01	0.56	0.10	-0.32	0.07	-0.27	-0.27	0.17
12	0.68	0.20	-0.27	0.38	0.31	0.87	-0.58	0.25	0.81	0.02	-0.38	-0.02	0.08	0.35	0.27	-0.08	-0.23	0.88	0.85	0.63
13	-0.07	-0.55	0.38	-0.44	0.30	0.33	-0.05	-0.02	0.52	0.25	0.29	0.62	0.33	0.10	0.70	0.29	0.19	0.42	0.53	0.53
14	-0.22	0.01	0.86	-0.04	0.72	-0.05	0.56	0.67	-0.11	0.34	0.70	0.72	0.55	0.56	0.77	0.01	0.63	-0.22	-0.13	0.50
15	0.25	0.67	0.24	0.47	0.43	0.02	0.32	0.56	-0.19	-0.18	0.33	0.00	0.58	0.40	0.14	-0.32	0.48	-0.20	-0.19	0.28
16	0.61	0.04	0.01	0.24	0.43	0.84	-0.42	0.31	0.79	0.21	-0.16	0.27	0.21	0.42	0.50	-0.07	-0.09	0.81	0.82	0.75
17	-0.56	-0.37	0.71	-0.43	0.19	-0.46	0.64	0.06	-0.41	0.33	0.62	0.71	0.27	0.11	0.44	0.25	0.45	-0.58	-0.43	0.02
18	0.04	-0.08	0.74	0.03	0.72	0.24	0.44	0.55	0.05	0.29	0.62	0.85	0.70	0.66	0.89	0.04	0.45	0.01	0.13	0.79
19	0.58	0.21	0.22	0.52	0.63	0.70	-0.08	0.70	0.45	0.31	-0.09	0.40	0.02	0.65	0.48	-0.34	-0.06	0.05	0.46	0.70
20	-0.05	-0.02	0.72	0.02	0.58	0.05	0.38	0.63	0.15	0.27	0.50	0.63	0.40	0.54	0.77	0.24	0.40	-0.05	0.15	0.56
21	1.00	0.45	-0.43	0.77	0.15	0.82	-0.51	0.34	0.66	-0.01	-0.62	-0.18	-0.20	0.36	-0.04	-0.35	-0.45	0.69	0.68	0.50
22	0.45	1.00	-0.24	0.76	0.20	0.14	-0.07	0.48	-0.08	-0.18	-0.24	-0.42	0.11	0.42	-0.23	-0.29	0.01	-0.01	-0.07	0.11
23	-0.43	-0.24	1.00	-0.28	0.60	-0.28	0.76	0.44	-0.31	0.34	0.85	0.82	0.53	0.40	0.75	0.07	0.68	-0.45	-0.32	0.35
24	0.77	0.76	-0.28	1.00	0.27	0.50	-0.20	0.60	0.21	-0.03	-0.45	-0.25	-0.11	0.54	-0.10	-0.38	-0.28	0.28	0.24	0.34
25	0.15	0.20	0.60	0.27	1.00	0.30	0.47	0.66	0.09	0.27	0.45	0.59	0.54	0.73	0.73	-0.09	0.42	0.05	0.13	0.68
26	0.82	0.14	-0.28	0.50	0.30	1.00	-0.57	0.30	0.74	0.07	-0.46	0.08	-0.05	0.44	0.25	-0.26	-0.42	0.89	0.82	0.70
27	-0.51	-0.07	0.76	-0.20	0.47	-0.57	1.00	0.28	-0.59	0.18	0.80	0.52	0.46	0.14	0.37	0.04	0.72	-0.74	-0.62	-0.02
28	0.34	0.48	0.44	0.60	0.66	0.30	0.28	1.00	-0.01	0.22	0.21	0.39	0.29	0.77	0.50	-0.29	0.19	0.00	0.02	0.61
29	0.66	-0.08	-0.31	0.21	0.09	0.74	-0.59	-0.01	1.00	0.14	-0.49	0.02	-0.20	0.08	0.20	0.16	-0.35	0.84	0.96	0.45
30	-0.01	-0.18	0.34	-0.03	0.27	0.07	0.18	0.22	0.14	1.00	0.15	0.43	0.01	0.19	0.34	0.13	0.10	0.02	0.12	0.25
31	-0.62	-0.24	0.85	-0.45	0.45	-0.46	0.80	0.21	-0.49	0.15	1.00	0.63	0.75	0.17	0.56	0.10	0.82	-0.57	-0.50	0.14
32	-0.18	-0.42	0.82	-0.25	0.59	0.08	0.52	0.39	0.02	0.43	0.63	1.00	0.44	0.44	0.86	0.10	0.44	-0.10	0.05	0.59
33	-0.20	0.11	0.53	-0.11	0.54	-0.05	0.46	0.29	-0.20	0.01	0.75	0.44	1.00	0.36	0.56	0.09	0.71	-0.16	-0.13	0.45
34	0.36	0.42	0.40	0.54	0.73	0.44	0.14	0.77	0.08	0.19	0.17	0.44	0.36	1.00	0.60	-0.12	0.11	0.16	0.17	0.74
35	-0.04	-0.23	0.75	-0.10	0.73	0.25	0.37	0.50	0.20	0.34	0.56	0.86	0.56	0.60	1.00	0.15	0.39	0.11	0.25	0.75
36	-0.35	-0.29	0.07	-0.38	-0.09	-0.26	0.04	-0.29	0.16	0.13	0.10	0.10	0.09	-0.12	0.15	1.00	0.14	-0.11	0.15	-0.14
37	-0.45	0.01	0.68	-0.28	0.42	-0.42	0.72	0.19	-0.35	0.10	0.82	0.44	0.71	0.11	0.39	0.14	1.00	-0.52	-0.41	0.07
38	0.69	-0.01	-0.45	0.28	0.05	0.89	-0.74	0.00	0.84	0.02	-0.57	-0.10	-0.16	0.16	0.11	-0.11	-0.52	1.00	0.91	0.52
39	0.68	-0.07	-0.32	0.24	0.13	0.82	-0.62	0.02	0.96	0.12	-0.50	0.05	-0.13	0.17	0.25	0.15	-0.41	0.91	1.00	0.56
40	0.50	0.11	0.35	0.34	0.68	0.70	-0.02	0.61	0.45	0.25	0.14	0.59	0.45	0.74	0.75	-0.14	0.07	0.52	0.56	1.00
41	0.47	-0.07	0.01	0.18	0.44	0.79	-0.40	0.31	0.74	0.19	-0.18	0.28	0.11	0.45	0.53	0.08	-0.15	0.79	0.80	0.69
42	0.70	0.33	0.01	0.52	0.50	0.81	-0.37	0.59	0.67	0.11	-0.21	0.19	0.20	0.62	0.47	-0.10	-0.12	0.70	0.73	0.79
43	-0.29	-0.28	0.76	-0.37	0.57	-0.01	0.43	0.25	0.11	0.32	0.70	0.76	0.61	0.30	0.76	0.22	0.70	-0.07	0.07	0.46
44	-0.04	0.49	0.54	0.22	0.51	-0.10	0.40	0.59	-0.19	0.07	0.49	0.32	0.59	0.54	0.41	0.04	0.62	-0.30	-0.22	0.32
45	0.83	0.48	-0.11	0.70	0.42	0.80	-0.36	0.58	0.57	0.05	-0.32	0.03	0.08	0.57	0.22	-0.33	-0.15	0.62	0.60	0.67
46	0.39	0.35	0.28	0.35	0.62	0.44	0.00	0.57	0.51	0.09	0.14	0.28	0.42	0.55	0.59	0.11	0.27	0.35	0.51	0.66
47	0.81	0.40	-0.20	0.62	0.44	0.91	-0.44	0.44	0.58	0.01	-0.33	0.05	0.15	0.60	0.27	-0.34	-0.22	0.75	0.66	0.73
48	0.75	0.15	-0.14	0.47	0.37	0.89	-0.46	0.41	0.70	0.16	-0.36	0.15	-0.08	0.47	0.28	-0.30	-0.26	0.77	0.72	0.65
49	-0.23	0.07	0.80	0.03	0.59	-0.19	0.66	0.71	-0.27	0.35	0.64	0.63	0.44	0.51	0.63	0.06	0.54	-0.41	-0.28	0.34
50	-0.34	-0.22	0.81	-0.28	0.44	-0.23	0.56	0.34	-0.12	0.33	0.72	0.77	0.62	0.34	0.75	0.35	0.56	-0.30	-0.09	0.43
51	0.51	0.87	-0.17	0.84	0.19	0.20	-0.07	0.64	-0.08	-0.09	-0.28	-0.30	-0.01	0.51	-0.14	-0.27	-0.14	0.00	-0.04	0.18
52	0.12	0.03	0.63	0.15	0.74	0.35	0.21	0.70	0.29	0.30	0.40	0.65	0.45	0.67	0.85	0.06	0.31	0.21	0.32	0.75
53	0.52	0.42	0.30	0.55	0.67	0.48	0.21	0.66	0.07	0.13	0.16	0.34	0.34	0.66	0.39	-0.57	0.19	0.19	0.11	0.64
54	0.68	0.22	0.10	0.57	0.57	0.78	-0.12	0.56	0.40	0.23	-0.12	0.42	0.23	0.72	0.50	-0.27	-0.16	0.53	0.52	0.87
55	0.20	-0.07	0.65	0.07	0.74	0.46	0.21	0.64	0.33	0.31	0.43	0.77	0.51	0.65	0.89	-0.06	0.34	0.30	0.38	0.85
56	0.59	-0.05	-0.15	0.21	0.28	0.75	-0.46	0.12	0.89	0.13	-0.29	0.16	0.06	0.23	0.34	0.21	-0.14	0.75	0.88	0.57
57	-0.12	0.20	0.75	0.21	0.62	-0.18	0.68	0.71	-0.29	0.17	0.61	0.58	0.45	0.62	0.59	0.04	0.50	-0.47	-0.30	0.37
58	-0.08	0.15	0.81	0.10	0.71	-0.01	0.53	0.70	-0.10	0.33	0.61	0.75	0.59	0.68	0.80	0.10	0.53	-0.22	-0.06	0.63
59	0.64	0.64	0.06	0.77	0.51	0.53	-0.10	0.79	0.34	0.06	-0.13	0.07	0.24	0.72	0.35	-0.09	-0.03	0.31	0.38	0.62
60	0.44	-0.03	0.15	0.15	0.49	0.69	-0.26	0.37	0.75	0.25	-0.05	0.39	0.21	0.43	0.58	0.15	0.05	0.66	0.76	0.72

# Correlation ( $\rho_{ij}$ with $i=41\dots60, j=1\dots60$ )

	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
1	0.35	0.23	0.79	0.26	-0.06	0.35	-0.01	0.06	0.57	0.83	-0.40	0.68	0.16	0.30	0.77	0.19	0.47	0.71	0.04	0.43
2	0.35	0.16	-0.16	-0.46	0.21	0.01	0.24	0.38	-0.45	-0.35	-0.16	-0.19	-0.09	0.20	-0.12	0.49	-0.52	-0.37	-0.07	0.31
3	0.63	0.74	0.45	0.41	0.67	0.77	0.61	0.65	0.09	0.19	0.09	0.53	0.35	0.41	0.57	0.81	0.11	0.32	0.52	0.79
4	0.72	0.69	-0.17	-0.29	0.67	0.41	0.75	0.76	-0.47	-0.38	0.11	0.14	0.17	0.50	0.19	0.81	-0.44	-0.27	0.39	0.63
5	0.29	0.41	0.70	0.61	0.27	0.53	0.24	0.21	0.78	0.73	0.09	0.81	0.56	0.44	0.85	0.11	0.77	0.85	0.43	0.39
6	0.43	0.47	0.49	0.41	0.41	0.45	0.33	0.49	0.59	0.42	0.14	0.76	0.49	0.47	0.71	0.24	0.62	0.65	0.45	0.47
7	0.55	0.74	0.05	0.26	0.72	0.45	0.83	0.63	0.15	0.11	0.51	0.43	0.65	0.92	0.52	0.45	0.23	0.40	0.72	0.50
8	0.50	0.79	-0.18	0.26	0.85	0.58	0.84	0.65	-0.02	-0.22	0.71	0.30	0.60	0.65	0.29	0.50	0.08	0.11	0.84	0.45
9	-0.19	-0.05	0.62	0.65	-0.13	0.34	-0.23	-0.31	0.74	0.71	0.04	0.49	0.31	-0.01	0.47	-0.27	0.78	0.74	0.16	-0.02
10	0.12	-0.01	0.83	0.36	-0.27	0.23	-0.22	-0.19	0.55	0.86	-0.49	0.53	0.02	0.04	0.60	0.01	0.44	0.66	-0.15	0.23
11	-0.19	0.02	-0.13	0.28	0.20	0.06	0.15	0.05	0.35	0.00	0.58	0.14	0.52	0.39	0.07	-0.27	0.54	0.32	0.40	-0.20
12	0.89	0.88	0.13	0.03	0.77	0.67	0.85	0.83	-0.16	-0.17	0.17	0.47	0.33	0.55	0.50	0.85	-0.21	0.02	0.55	0.84
13	0.60	0.38	0.70	0.04	0.09	0.40	0.17	0.29	0.18	0.52	-0.55	0.55	-0.03	0.23	0.65	0.54	0.03	0.34	-0.02	0.62
14	0.30	0.35	0.79	0.69	0.20	0.58	0.09	0.15	0.83	0.70	0.07	0.85	0.43	0.21	0.79	0.07	0.74	0.86	0.36	0.44
15	-0.06	0.34	0.16	0.68	0.43	0.48	0.31	0.09	0.36	0.22	0.58	0.29	0.58	0.22	0.28	0.00	0.49	0.44	0.58	0.09
16	0.86	0.86	0.38	0.15	0.73	0.67	0.80	0.80	0.02	0.12	0.04	0.59	0.45	0.64	0.68	0.82	-0.05	0.25	0.51	0.85
17	-0.35	-0.39	0.51	0.27	-0.46	-0.18	-0.48	-0.43	0.52	0.69	-0.32	0.16	-0.03	-0.05	0.21	-0.36	0.49	0.54	-0.29	-0.21
18	0.34	0.44	0.69	0.50	0.26	0.50	0.31	0.23	0.63	0.80	-0.02	0.74	0.51	0.63	0.83	0.27	0.68	0.85	0.40	0.45
19	0.67	0.74	0.28	0.25	0.77	0.60	0.72	0.83	0.31	0.06	0.30	0.68	0.68	0.71	0.70	0.54	0.33	0.42	0.61	0.69
20	0.39	0.46	0.63	0.52	0.26	0.70	0.09	0.19	0.71	0.72	0.11	0.86	0.23	0.26	0.77	0.27	0.71	0.80	0.48	0.54
21	0.47	0.70	-0.29	-0.04	0.83	0.39	0.81	0.75	-0.23	-0.34	0.51	0.12	0.52	0.68	0.20	0.59	-0.12	-0.08	0.64	0.44
22	-0.07	0.33	-0.28	0.49	0.48	0.35	0.40	0.15	0.07	-0.22	0.87	0.03	0.42	0.22	-0.07	-0.05	0.20	0.15	0.64	-0.03
23	0.01	0.01	0.76	0.54	-0.11	0.28	-0.20	-0.14	0.80	0.81	-0.17	0.63	0.30	0.10	0.65	-0.15	0.75	0.81	0.06	0.15
24	0.18	0.52	-0.37	0.22	0.70	0.35	0.62	0.47	0.03	-0.28	0.84	0.15	0.55	0.57	0.07	0.21	0.21	0.10	0.77	0.15
25	0.44	0.50	0.57	0.51	0.42	0.62	0.44	0.37	0.59	0.44	0.19	0.74	0.67	0.57	0.74	0.28	0.62	0.71	0.51	0.49
26	0.79	0.81	-0.01	-0.10	0.80	0.44	0.91	0.89	-0.19	-0.23	0.20	0.35	0.48	0.78	0.46	0.75	-0.18	-0.01	0.53	0.69
27	-0.40	-0.37	0.43	0.40	-0.36	0.00	-0.44	-0.46	0.66	0.56	-0.07	0.21	0.21	-0.12	0.21	-0.46	0.68	0.53	-0.10	-0.26
28	0.31	0.59	0.25	0.59	0.58	0.57	0.44	0.41	0.71	0.34	0.64	0.70	0.66	0.56	0.64	0.12	0.71	0.70	0.79	0.37
29	0.74	0.67	0.11	-0.19	0.57	0.51	0.58	0.70	-0.27	-0.12	-0.08	0.29	0.07	0.40	0.33	0.89	-0.29	-0.10	0.34	0.75
30	0.19	0.11	0.32	0.07	0.05	0.09	0.01	0.16	0.35	0.33	-0.09	0.30	0.13	0.23	0.31	0.13	0.17	0.33	0.06	0.25
31	-0.18	-0.21	0.70	0.49	-0.32	0.14	-0.33	-0.36	0.64	0.72	-0.28	0.40	0.16	-0.12	0.43	-0.29	0.61	0.61	-0.13	-0.05
32	0.28	0.19	0.76	0.32	0.03	0.28	0.05	0.15	0.63	0.77	-0.30	0.65	0.34	0.42	0.77	0.16	0.58	0.75	0.07	0.39
33	0.11	0.20	0.61	0.59	0.08	0.42	0.15	-0.08	0.44	0.62	-0.01	0.45	0.34	0.23	0.51	0.06	0.45	0.59	0.24	0.21
34	0.45	0.62	0.30	0.54	0.57	0.55	0.60	0.47	0.51	0.34	0.51	0.67	0.66	0.72	0.65	0.23	0.62	0.68	0.72	0.43
35	0.53	0.47	0.76	0.41	0.22	0.59	0.27	0.28	0.63	0.75	-0.14	0.85	0.39	0.50	0.89	0.34	0.59	0.80	0.35	0.58
36	0.08	-0.10	0.22	0.04	-0.33	0.11	-0.34	-0.30	0.06	0.35	-0.27	0.06	-0.57	-0.27	-0.06	0.21	0.04	0.10	-0.09	0.15
37	-0.15	-0.12	0.70	0.62	-0.15	0.27	-0.22	-0.26	0.54	0.56	-0.14	0.31	0.19	-0.16	0.34	-0.14	0.50	0.53	-0.03	0.05
38	0.79	0.70	-0.07	-0.30	0.62	0.35	0.75	0.77	-0.41	-0.30	0.00	0.21	0.19	0.53	0.30	0.75	-0.47	-0.22	0.31	0.66
39	0.80	0.73	0.07	-0.22	0.60	0.51	0.66	0.72	-0.28	-0.09	-0.04	0.32	0.11	0.52	0.38	0.88	-0.30	-0.06	0.38	0.76
40	0.69	0.79	0.46	0.32	0.67	0.66	0.73	0.65	0.34	0.43	0.18	0.75	0.64	0.87	0.85	0.57	0.37	0.63	0.62	0.72
41	1.00	0.83	0.32	0.03	0.63	0.64	0.73	0.79	0.07	0.04	0.00	0.64	0.29	0.55	0.68	0.80	-0.06	0.19	0.47	0.89
42	0.83	1.00	0.24	0.32	0.86	0.80	0.85	0.80	0.19	0.11	0.40	0.69	0.52	0.69	0.70	0.77	0.17	0.38	0.81	0.84
43	0.32	0.24	1.00	0.54	0.07	0.50	0.04	0.11	0.55	0.71	-0.34	0.68	0.25	0.13	0.72	0.25	0.45	0.69	0.06	0.49
44	0.03	0.32	0.54	1.00	0.30	0.56	0.17	0.01	0.62	0.52	0.46	0.52	0.46	0.14	0.45	-0.02	0.66	0.73	0.52	0.20
45	0.63	0.86	0.07	0.30	1.00	0.67	0.89	0.85	0.02	-0.12	0.53	0.47	0.68	0.69	0.51	0.63	0.10	0.22	0.76	0.66
46	0.64	0.80	0.50	0.56	0.67	1.00	0.55	0.52	0.36	0.37	0.31	0.79	0.43	0.43	0.71	0.63	0.41	0.59	0.73	0.76
47	0.73	0.85	0.04	0.17	0.89	0.55	1.00	0.87	-0.10	-0.19	0.40	0.41	0.66	0.80	0.50	0.66	-0.04	0.13	0.66	0.66
48	0.79	0.80	0.11	0.01	0.85	0.52	0.87	1.00	-0.08	-0.19	0.20	0.46	0.55	0.68	0.55	0.75	-0.09	0.08	0.51	0.76
49	0.07	0.19	0.55	0.62	0.02	0.36	-0.10	-0.08	1.00	0.70	0.21	0.65	0.32	0.15	0.61	-0.14	0.81	0.80	0.38	0.18
50	0.04	0.11	0.71	0.52	-0.12	0.37	-0.19	-0.19	0.70	1.00	-0.16	0.60	0.09	0.16	0.60	0.03	0.67	0.82	0.16	0.22
51	0.00	0.40	-0.34	0.46	0.53	0.31	0.40	0.20	0.21	-0.16	1.00	0.13	0.45	0.31	0.01	-0.07	0.30	0.21	0.75	0.00
52	0.64	0.69	0.68	0.52	0.47	0.79	0.41	0.46	0.65	0.60	0.13	1.00	0.44	0.48	0.91	0.45	0.63	0.77	0.60	0.72
53	0.29	0.52	0.25	0.46	0.68	0.43	0.66	0.55	0.32	0.09	0.45	0.44	1.00	0.70	0.57	0.16	0.38	0.46	0.53	0.32
54	0.55	0.69	0.13	0.14	0.69	0.43	0.80	0.68	0.15	0.16	0.31	0.48	0.70	1.00	0.60	0.49	0.25	0.42	0.60	0.52
55	0.68	0.70	0.72	0.45	0.51	0.71	0.50	0.55	0.61	0.60	0.01	0.91	0.57	0.60	1.00	0.48	0.52	0.75	0.49	0.76
56	0.80	0.77	0.25	-0.02	0.63	0.63	0.66	0.75	-0.14	0.03	-0.07	0.45	0.16	0.49	0.48	1.00	-0.10	0.07	0.44	0.87
57	-0.06	0.17	0.45	0.66	0.10	0.41	-0.04	-0.09	0.81	0.67	0.30	0.63	0.38	0.25	0.52	-0.10	1.00	0.80	0.48	0.09
58	0.19	0.38	0.69	0.73	0.22	0.59	0.13	0.08	0.80	0.82	0.21	0.77	0.46	0.42	0.75	0.07	0.80	1.00	0.48	0.36
59	0.47	0.81	0.06	0.52	0.76	0.73	0.66	0.51	0.38	0.16	0.75	0.60	0.53	0.60	0.49	0.44	0.48	0.48	1.00	0.50
60	0.89	0.84	0.49	0.20	0.66	0.76	0.66	0.76	0.18	0.22	0.00	0.72	0.32	0.52	0.76	0.87	0.09	0.36	0.50	1.00

## Appendix B – Case Study (Risk-Adjusted & Time-Adjusted): Modified Zeus Ltd.

This is the same problem as described previously, except that the expected return ( $\mu$ ) is no longer a constant value.

$$\mu_i(t) = \frac{(a_i \cdot t^2 + b_i \cdot t + c_i) \cdot \lambda_{\{t \leq T_i\}} + (a_i \cdot T_i^2 + b_i \cdot T_i + c_i) \cdot \lambda_{\{t > T_i\}}}{(1 + d)^t}$$

As noted, the expected return is characterized by

$$a_i \text{ with } i = 1 \dots 60;$$

$$b_i \text{ with } i = 1 \dots 60;$$

$$c_i \text{ with } i = 1 \dots 60;$$

$$T_i \text{ with } i = 1 \dots 60;$$

Zeus Ltd. claims that the risk free interest rate is 1.0% per annum, and hence  $d = 1.0\%$

The other parameters are given in the tables of the following pages.



	ai	bi	ci	Ti
1	5.33	-7.60	0	3.85
2	2.46	-3.68	0	5.32
3	2.44	-2.25	0	5.71
4	1.48	-1.91	0	6.48
5	1.40	-1.84	0	6.67
6	0.88	-0.98	0	8.11
7	2.26	-0.13	0	6.23
8	2.06	-2.70	0	8.33
9	8.24	-11.33	0	3.39
10	1.06	-0.26	0	6.48
11	0.74	-0.80	0	8.75
12	13.58	-15.82	0	3.69
13	3.49	-4.52	0	4.37
14	0.84	-0.01	0	7.74
15	0.83	-0.89	0	8.31
16	2.08	-1.71	0	5.32
17	0.86	-0.64	0	6.26
18	1.59	-1.39	0	5.80
19	1.94	-2.66	0	5.81
20	0.76	-1.00	0	8.76
21	7.69	-10.78	0	3.44
22	0.94	-0.01	0	7.32
23	4.26	-4.41	0	4.11
24	3.16	-1.98	0	5.12
25	14.38	-22.58	0	2.71
26	1.92	-0.68	0	5.60
27	9.70	-14.36	0	3.12
28	38.38	-66.14	0	2.31
29	91.15	-142.54	0	2.44
30	15.21	-25.57	0	2.50
31	0.81	0.64	0	8.85
32	4.43	-4.24	0	4.97
33	0.97	-0.28	0	7.35
34	2.58	-3.82	0	5.42
35	2.26	-2.66	0	5.34
36	4.25	-6.32	0	4.25
37	0.91	0.12	0	7.37
38	2.33	-2.75	0	4.49
39	0.83	0.35	0	7.78
40	1.54	-1.48	0	6.05
41	16.23	-23.18	0	3.26
42	1.60	-0.40	0	6.02
43	8.73	-13.03	0	3.25
44	6.13	-7.61	0	3.54
45	1.10	-0.77	0	8.69
46	4.01	-4.83	0	4.44
47	1.12	1.00	0	8.11
48	34.09	-51.83	0	2.62
49	4.13	-3.83	0	4.24
50	0.56	-0.50	0	8.33
51	2.10	-1.64	0	5.41
52	1.28	-1.09	0	6.66
53	1.17	-0.05	0	6.61
54	32.81	-52.14	0	2.43
55	55.22	-87.48	0	2.03
56	1.96	-2.08	0	5.58
57	0.86	-0.75	0	8.05
58	44.20	-71.20	0	2.14
59	2.80	0.05	0	6.14
60	1.75	0.47	0	8.19

## Appendix C – Exhaustive Search Algorithm (Exact)

```
//Algorithm - Exhaustive Search
function [Ind,u] = exact(mu,sigma,corr,r)
% exact algorithm(mu,sigma,corr,r)

size=length(mu);
Ind=zeros(1,size);
Ind(1)=1;           %initialize Ind = 1000...000
u=utility(mu,sigma,corr,r,Ind); %initilize utility(1000...000)

%from 100...001 to 111...111
for i=2.^(size-1)+1:2.^size-1
    Ind_temp=toArray(i,size);
    u_temp=utility(mu,sigma,corr,r,Ind_temp);
    if (u_temp>u)
        Ind=Ind_temp;
        u=u_temp;
    end
end
end
```

## Appendix D – CPLEX Optimization (Exact)

```
//CPLEX Optimization
function [Ind,u] = cpl(mu,sigma,corr,r)
% cplex optimization(mu,sigma,corr,r)

f=(-r.*mu)';

sizeProblem=length(sigma);
sigmaMatrix=zeros(sizeProblem,sizeProblem);

for i=1:sizeProblem
    sigmaMatrix(i,i)=sigma(i);
end

H=r.^2.*(sigmaMatrix*corr*sigmaMatrix);

% to make sure H is symmetric
for i=1:sizeProblem
    for j=i:sizeProblem
        if H(i,j)~=H(j,i)
            H(i,j)=H(j,i);
        end
    end
end

Aineq=[];
bineq=[];

Aeq=zeros(1,sizeProblem);
Aeq(1)=1;
beq=1;

lb=[];
ub=[];

ctype='';
for i=1:sizeProblem
    ctype=strcat(ctype,'B');
```

```
end
```

```
options = cplexoptimset('Display','off');
```

```
options.Diagnostics = 'on';
```

```
[x, fval, exitflag, output] = cplexmiqp (H, f, Aineq, bineq, Aeq,  
beq,...
```

```
[], [], [], lb, ub, ctype, [], options);
```

```
Ind=x';
```

```
u=-fval;
```

## Appendix E – Greedy Algorithm

```
//Algorithm - Greedy
function [Ind,u] = greedy(mu,sigma,corr,r,Ind)
% greedy algorithm(mu,sigma,corr,r)

size=length(mu);
if (nargin==4)          %if Ind is not provided, to initialize Ind
    Ind=zeros(1,size);
    Ind(1)=1;          %initialize Ind = 1000...000
elseif (nargin==5)
    ;
else
    disp('error: nargin is INVALID');
    return;
end

u=utility(mu,sigma,corr,r,Ind); %initilize utility(1000...000)

for i=0:size-2
    Ind(size-i)=1-Ind(size-i);
    u_temp=utility(mu,sigma,corr,r,Ind);
    if(u_temp>u)
        u=u_temp;
    else
        Ind(size-i)=1-Ind(size-i);
    end
end
end
```

## Appendix F – Hill Climbing Algorithm

```
//Algorithm - Hill Climbing
function [Ind,u] = hclimb(mu,sigma,corr,r,opt,Ind)
% hclimb algorithm(mu,sigma,corr,r)
% OR
% hclimb algorithm(mu,sigma,corr,r,Ind)
if (nargin==4)
    opt=4;                %initialize opt = 4
    size=length(mu);
    Ind=zeros(1,size);
    Ind(1)=1;             %initialize Ind = 1000...000
elseif (nargin==5)
    size=length(mu);
    Ind=zeros(1,size);
    Ind(1)=1;             %initialize Ind = 1000...000
elseif (nargin==6)
    ;
else
    disp('error: nargin is INVALID');
    return;
end

u=utility(mu,sigma,corr,r,Ind); %initilize utility(1000...000)

flag1=true;
flag2=true;
flag3=true;
flag4=true;

while flag1||flag2||flag3||flag4

    flag1=false;
    flag2=false;
    flag3=false;
    flag4=false;
    %one step each time

    [Ind_temp,u_temp]=onestep(mu,sigma,corr,r,Ind);
```

```

if u_temp>u
    u=u_temp;
    Ind=Ind_temp;
    flag1=true;
end

%two steps each time
if opt >= 2
    [Ind_temp,u_temp]=twosteps(mu,sigma,corr,r,Ind);
    if u_temp>u
        u=u_temp;
        Ind=Ind_temp;
        flag2=true;
    end
else
    continue;
end

%three steps each time
if opt >= 3
    [Ind_temp,u_temp]=threesteps(mu,sigma,corr,r,Ind);
    if u_temp>u
        u=u_temp;
        Ind=Ind_temp;
        flag3=true;
    end
else
    continue;
end

%four steps each time
if opt >= 4
    [Ind_temp,u_temp]=foursteps(mu,sigma,corr,r,Ind);
    if u_temp>u
        u=u_temp;
        Ind=Ind_temp;
        flag4=true;
    end
else
    continue;
end

```

```

        end

    end

//Module 1 - One step a time
function [Ind,u] = onestep(mu,sigma,corr,r,Ind)
%onestep(mu,sigma,corr,r,Ind)

[Ind,u]=greedy(mu,sigma,corr,r,Ind);

//Module 2 - Two steps a time
function [Ind,u] = twosteps(mu,sigma,corr,r,Ind)
%onestep(mu,sigma,corr,r,Ind)

size=length(mu);
u=utility(mu,sigma,corr,r,Ind);

for i=0:size-2
    for j=i+1:size-2
        Ind(size-i)=1-Ind(size-i);
        Ind(size-j)=1-Ind(size-j);
        u_temp=utility(mu,sigma,corr,r,Ind);
        if(u_temp>u)
            u=u_temp;
        else
            Ind(size-i)=1-Ind(size-i);
            Ind(size-j)=1-Ind(size-j);
        end
    end
end

//Module 3 - Three steps a time
function [Ind,u] = threesteps(mu,sigma,corr,r,Ind)
%onestep(mu,sigma,corr,r,Ind)

size=length(mu);
u=utility(mu,sigma,corr,r,Ind);

for i=0:size-2

```



```

    for j=i+1:size-2
        for m=j+1:size-2
            Ind(size-i)=1-Ind(size-i);
            Ind(size-j)=1-Ind(size-j);
            Ind(size-m)=1-Ind(size-m);
            u_temp=utility(mu,sigma,corr,r,Ind);
            if(u_temp>u)
                u=u_temp;
            else
                Ind(size-i)=1-Ind(size-i);
                Ind(size-j)=1-Ind(size-j);
                Ind(size-m)=1-Ind(size-m);
            end
        end
    end
end

//Module 4 - Four steps a time
function [Ind,u] = foursteps(mu,sigma,corr,r,Ind)
%onestep(mu,sigma,corr,r,Ind)

size=length(mu);
u=utility(mu,sigma,corr,r,Ind);

for i=0:size-2
    for j=i+1:size-2
        for m=j+1:size-2
            for n=m+1:size-2
                Ind(size-i)=1-Ind(size-i);
                Ind(size-j)=1-Ind(size-j);
                Ind(size-m)=1-Ind(size-m);
                Ind(size-n)=1-Ind(size-n);
                u_temp=utility(mu,sigma,corr,r,Ind);
                if(u_temp>u)
                    u=u_temp;
                else
                    Ind(size-i)=1-Ind(size-i);
                    Ind(size-j)=1-Ind(size-j);
                    Ind(size-m)=1-Ind(size-m);
                    Ind(size-n)=1-Ind(size-n);
                end
            end
        end
    end
end

```

```
end
end
end
end
end
```

## Appendix G – Random Restart Climbing Algorithm

```
//Algorithm - Random Restart Hill Climbing
function [Ind,u] = hclimb_rand(mu,sigma,corr,r,opt,n)
% hclimb_rand algorithm(mu,sigma,corr,r)
%- default n=50
% OR
% hclimb_rand algorithm(mu,sigma,corr,r,n)
%- to specify number of iterations

size=length(mu);

if (nargin==4)
    opt=2;           %by default, 2-opt
    n=5;             %by default, ranmization 5 times
elseif (nargin==5)
    n=5;             %by default, ranmization 5 times
elseif (nargin==6)
    ;
else
    disp('ERROR! Nargin is INVALID!');
    return;
end

u=-inf;

for i=1:n
    Ind_start=Ind_rand(size);
    [Ind_temp,u_temp]=hclimb(mu,sigma,corr,r,opt,Ind_start);
    if (u_temp>u)
        Ind=Ind_temp;
        u=u_temp;
    end
end

//Module 1 - Random Restart
function Ind = Ind_rand(size)
% Ind_rand(size)
```

```
Ind=rand(1,size);  
for i=1:size;  
    Ind(i)=Ind(i)>0.5;  
end  
Ind(1)=1;
```

## Appendix H – Stochastic Gradient Ascent Algorithm

```
//Algorithm - Stochastic Gradient Ascent
function [Ind,u] = stochgrad(mu,sigma,corr,r,opt)
% hclimb algorithm(mu,sigma,corr,r)
% OR
% hclimb algorithm(mu,sigma,corr,r,Ind)
if (nargin==4)
    opt=4;                %initialize opt = 4
    size=length(mu);
    Ind=zeros(1,size);
    Ind(1)=1;             %initialize Ind = 1000...000
elseif (nargin==5)
    size=length(mu);
    Ind=zeros(1,size);
    Ind(1)=1;             %initialize Ind = 1000...000
else
    disp('error: nargin is INVALID');
    return;
end

u=utility(mu,sigma,corr,r,Ind); %initilize utility(1000...000)

flag1=true;
flag2=true;
flag3=true;
flag4=true;

while flag1||flag2||flag3||flag4

    flag1=false;
    flag2=false;
    flag3=false;
    flag4=false;

    if (rand()<0.1)                % 20% chance random gradient
        disp(Ind);
        Ind=stochgrad_rand(Ind,opt);
        disp(Ind);
    end
end
```

```

        flag1=true;
        continue;
    else
        %one step each time
        [Ind_temp,u_temp]=onestep(mu,sigma,corr,r,Ind);
        if u_temp>u
            u=u_temp;
            Ind=Ind_temp;
            flag1=true;
        end

        %two steps each time
        if opt >= 2
            [Ind_temp,u_temp]=twosteps(mu,sigma,corr,r,Ind);
            if u_temp>u
                u=u_temp;
                Ind=Ind_temp;
                flag2=true;
            end
        else
            continue;
        end

        %three steps each time
        if opt >= 3
            [Ind_temp,u_temp]=threesteps(mu,sigma,corr,r,Ind);
            if u_temp>u
                u=u_temp;
                Ind=Ind_temp;
                flag3=true;
            end
        else
            continue;
        end

        %four steps each time
        if opt >= 4
            [Ind_temp,u_temp]=foursteps(mu,sigma,corr,r,Ind);
            if u_temp>u
                u=u_temp;
            end
        end
    end
end

```

```

        Ind=Ind_temp;
        flag4=true;
    end
else
    continue;
end
end
end

//Module 1 - Random Gradient
function Ind=stochgrad_rand(Ind,opt)

for i=1:opt
    position_rand=round(rand()*(length(Ind)-2))+2;
    Ind(position_rand)=1-Ind(position_rand);
end

```

## Appendix I – Find Optimal Time

```
//Algorithm - Final Optimal Time (Non-Linear Programming)
function [v,utility] = findTime(a,b,sigma,corr,r,ind,discount,T,t)
% [v,utility] = findTime(a,b,sigma,corr,r,ind,discount,T,t)

t0 = t; % starting search point
    options = optimset('Display','off','Algorithm','active-
set','LargeScale','off','MaxFunEvals',1e5,'MaxIter',1e6,'TolFun',1e-
8,'TolX',1e-8);
[v, fval] = fmincon(@(t)-
funobj(a,b,ind,discount,T,t),t0,[],[],[],[],[],[],@funcon,options);
fval=-fval;

utility=r*fval-utilityB(sigma,corr,r,ind);

//Module 1 - Constraint Function
function [c,ceq] = funcon(t)

c = -t;
ceq=[];

//Module 2 - Objective Function
function f = funobj (a,b,ind,discount,T,t)

m=length(t);
n=length(T);

Tind=zeros(m,n);
for i=1:m
    for j=1:n
        Tind(i,j)=(t(i)<T(j));
    end
end

f=((Tind.*a)'*(t.^2)+((1-Tind).*a.*T.^2)')...
+(Tind.*b)'*t+((1-Tind).*b.*T)')'...
./(1+discount).^t*ind';
```



## **Appendix J – Probability to Attain Global Optimum (2-opt Hill Climbing Algorithm)**

To solve the Risk-adjusted Multiple Investment Decision problem with different sizes  $N = 10 \dots 60$ , there are 100 sets of computer generated data for each problem size, and hence a total of 5,100 sets of data.

Using the 2-opt Hill Climbing Algorithm, each problem size is solved for 100 times and compare its results with the true global optimums.

Below is a table summarizing its probability to attain global optimum, which is a function with decreasing trend of problem size.

Problem Size	Probability to Attain Global Optimum
10	98.0%
11	98.0%
12	97.0%
13	96.0%
14	94.0%
15	95.0%
16	93.0%
17	92.0%
18	90.0%
19	89.0%
20	86.0%
21	86.0%
22	85.0%
23	83.0%
24	83.0%
25	80.0%
26	83.0%
27	79.0%
28	80.0%
29	78.0%
30	77.0%
31	78.0%
32	75.0%
33	76.0%
34	74.0%
35	72.0%
36	75.0%
37	70.0%
38	73.0%
39	72.0%
40	70.0%
41	70.0%
42	71.0%
43	69.0%
44	67.0%
45	68.0%
46	67.0%
47	67.0%
48	64.0%
49	67.0%
50	65.0%
51	63.0%
52	66.0%
53	62.0%
54	62.0%
55	61.0%
56	60.0%
57	59.0%
58	62.0%
59	59.0%
60	63.0%